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March 2017 | ₹ 35.00

MATHEMATICS Spectrum

TARGET 2017

Rapid Concept Revision

Hyperbola
Trigonometry

FROM CLASS XIth SYLLABUS

Vector
3-D
Probability

FROM CLASS XIIth SYLLABUS

TEST RIDER FOR JEE
TEST DRIVE FOR BOARD

JUVENILEZ

GOLDEN OLDIES *Conic Section*
RAINBOW *Applications of Derivatives*
FUN ARCADE

MEGA ISSUE 4
More Pages
More Features
More Practice...

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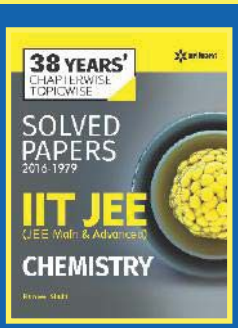
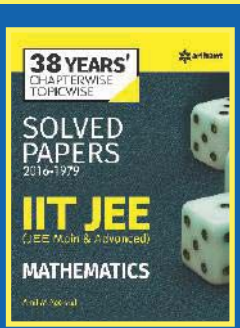
SOLVED PAPERS & PRACTICE SETS FOR

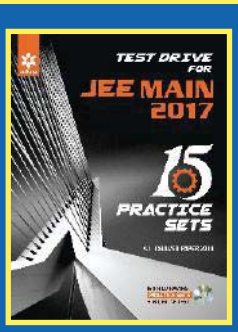
JEE MAIN & ADVANCED

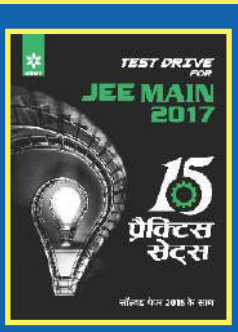
14 Years' SOLVED PAPERS JEE MAIN

38 Years' SOLVED PAPERS (JEE MAIN & ADVANCED)









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Issue 3 **Vol 4**

Dear Aspirants,

By now your studies would be in full swing as March is the month of examinations. New situations always pose new challenges and the same is applicable to March 2017. The difficulty level is a bit higher for aspirants this year due to elections. The decreased gap between JEE Mains and Board exams may dismantle your planning and affect your result.

Examinations are not only meant to test our knowledge and study skills. Rather, our personality as an individual is also tested through them. During an examination, our ability to do work, to deal with difficult situations, to withstand mental pressures and our thinking under stress are also tested. The day of examination is not at all a favourite for many among us, but nobody can avoid it. You may feel nervous, tense, scared, anxious or all on this particular day. But, remember! "Exams are not hurdles rather these are the stepping stones to success". So leave all your negative feeling, be relaxed and take it easy. Keep following points in mind to perform better in Boards:

- Careful reading of question paper is very important. In Boards, a proper time (e.g. half an hour in CBSE) is allotted for this task. Try to utilise this time properly. This will be the only procedure through which you can ignite your mind for the mammoth task ahead in next 3 hours.
- Prioritise and plan your paper also within this allotted time only i.e. don't waste this time in anything else.
- Try to convert those questions into flow charts or diagrams which are not clear to you. This will help in better visualisation of problems.
- Answer your best question first. Try to highlight the important parts of your answer. These will be helpful in catching the examiners attention at the time of checking of your response sheet.
- Try to be as neat and clean as possible. Start every new question from a fresh page.
- If you are short of time, avoid calculations and if there is some time left, check your answers again.
- Once your examination is over and you are out of the hall, avoid discussing the examination paper or questions with other candidates. This may create unnecessary tension.
- Last but not the least, "Try to stay positive and never let negative thoughts influence you at all".

Try to be enthusiastic while solving a problem because

"Nothing great was ever achieved without enthusiasm"

The spectrum team will be anxiously waiting for your feedbacks and comments.

Good luck

Sharma

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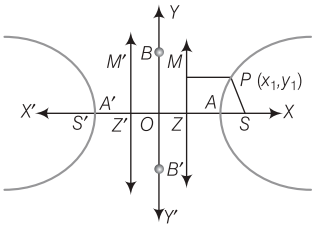
@ CLASS XI SYLLABUS

Rapid CONCEPT REVISION

TARGET 2017

HYPERBOLA

A hyperbola is the locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point (i.e. focus) and a fixed line (i.e. directrix) is always constant, which is greater than one. This ratio is called eccentricity (e).



Let S and MZ be the focus and directrix of the hyperbola.

If $P(x_1, y_1)$ is any point on the hyperbola, then

$$\frac{PS}{PM} = e \text{ or } PS = ePM,\ e > 1$$

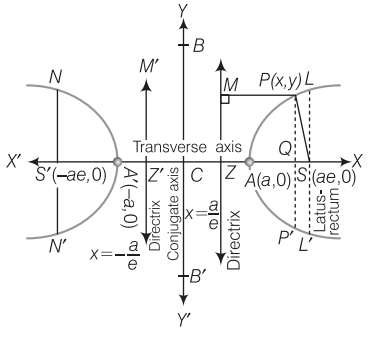
Or

A hyperbola is the locus of a point which moves in such a way that the difference of its distance from two fixed points (i.e. foci) is always constant.

Standard Equation of Hyperbola

Standard equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where, a and b are constants and $b^2 = a^2(e^2 - 1)$.

The sketch of the locus of a moving point which satisfy the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is shown in the adjoining figure.



Some Important Terms

(i) **Symmetry** On replacing y by $(-y)$ and x by $(-x)$, then the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ remains same. So, the curve is symmetrical about both X and Y-axes.

(ii) **Foci** The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the hyperbola and the distance between foci is $SS' = 2ae$.

(iii) **Directrices** The lines ZM and $Z'M'$ are called the directrices of the hyperbola and its equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$. Also, distance between the directrices is $ZZ' = \frac{2a}{e}$.

(iv) **Axes** The lines AA' and BB' are known as transverse axis and conjugate axis, respectively of the hyperbola.
(a) The length of the transverse axis = $AA' = 2a$
(b) The length of the conjugate axis = $BB' = 2b$

(v) **Centre** The point of intersection of the axes of the hyperbola is known as centre (C) of the hyperbola.

Note All chords passing through C are bisected at C.

(vi) **Vertices** The end points A and A' of transverse axis are known as the vertices of a hyperbola.
The coordinates of A and A' are $(a, 0)$ and $(-a, 0)$, respectively.

(vii) **Focal chord** A chord of a hyperbola passing through its focus is known as focal chord.

(viii) **Focal distance** The distance of any point on the hyperbola from the focus is called focal distance.
The difference of the focal distances of any point on

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the hyperbola is always constant and this constant is always equal to the length of the transverse axis of the hyperbola.

If P is any point of the hyperbola, then $S'P - SP = 2a$ = Length of transverse axis

(ix) **Latusrectum** The focal chord perpendicular to the transverse axis is called latusrectum.

If LL' and NN' are the latusrectums of a hyperbola, then these lines are perpendicular to the transverse axis AA' and passing through S and S', respectively.

Here, $L \equiv \left(ae, \frac{b^2}{a}\right)$, $L' \equiv \left(ae, \frac{-b^2}{a}\right)$, $N \equiv \left(-ae, \frac{b^2}{a}\right)$ and $N' \equiv \left(-ae, \frac{-b^2}{a}\right)$

and length of latusrectum $= LL' = \frac{2b^2}{a} = NN'$

(x) **Ordinate and double ordinate** Let P be a point on the hyperbola. From P, draw PQ perpendicular to the axis of the hyperbola and produced it to meet the curve again at P'. Then, PQ is called ordinate and PP' is called double ordinate.

(xi) **Eccentricity of hyperbola** The eccentricity of a hyperbola is $e = \sqrt{1 + (b^2/a^2)}$.

(xii) **Auxiliary circle** Equation of auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$.

(xiii) **Parametric equations of the hyperbola** The parametric equations of the hyperbola are $x = a\sec\theta$ and $y = b\tan\theta$, where θ is the parameter and $0 < \theta < 2\pi$.

(xiv) **Equation of chord** The equation of chord joining the points $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ is $\begin{vmatrix} x & y & 1 \\ a\sec\theta_1 & b\tan\theta_1 & 1 \\ a\sec\theta_2 & b\tan\theta_2 & 1 \end{vmatrix} = 0$

or $\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$.

Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola, is called the conjugate hyperbola of the given hyperbola. The conjugate hyperbola of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, i.e. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Properties of Hyperbola and Conjugate Hyperbola

Properties of hyperbola and conjugate hyperbola are given below :

	Hyperbola	Conjugate hyperbola
Standard equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{-x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	$(0, 0)$	$(0, 0)$
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$	$e = \sqrt{\frac{a^2 + b^2}{b^2}}$
Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parameter coordinates	$(a\sec\theta, b\tan\theta)$	$(a\tan\theta, b\sec\theta)$
Focal distances of point (x_1, y_1)	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal distances $(S'P - SP)$	$2a$	$2b$
Tangents at the vertices	$x = \pm a$	$y = \pm b$

Position of a Point with Respect to a Hyperbola

Let an equation of the hyperbola be $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$

If value of a hyperbola at (x_1, y_1) is $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$, then the following cases arise.

(i) If $S_1 = 0$, then the point lies on the hyperbola.
(ii) If $S_1 < 0$, then the point lies inside the hyperbola.
(iii) If $S_1 > 0$, then the point lies outside the hyperbola.

Various Form of the Equation of Tangent

1. Point Form

If $P(x_1, y_1)$ is any point on the hyperbola, then the equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

2. Parametric Form

If $P(a\sec\theta, b\tan\theta)$ is any point on the hyperbola, then the equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

3. Slope Form

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

Condition for Tangency and Points of Contact

A line $y = mx + c$ is said to be tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ iff $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are

$$\left(\pm \frac{a^2 m}{\sqrt{a^2m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2 - b^2}} \right).$$

Nature of Tangents Drawn from a Point to a Hyperbola

Two tangents can be drawn from a point to a hyperbola. The nature of tangents depends on the position of a point, which are given below :

(i) If a point lies outside, then two tangents are real and distinct.

(ii) If a point lies on the hyperbola, then two tangents are coincident.

(iii) If a point lies inside the hyperbola, then two tangents are imaginary.

Various Form of the Equation of Normal

1. Point Form

The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

2. Parametric Form

The equations of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(a \sec\theta, b \tan\theta)$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$.

3. Slope Form

The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope m is $y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$.

The coordinates of points of contact are

$$\left(\pm \frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp \frac{mb^2}{\sqrt{a^2 - b^2m^2}} \right).$$

Note Four normals can be drawn from any point to a hyperbola.

Conormal Points

Any four points on the hyperbola at which the normals pass through a common point, are called conormal points.

Note Tangent drawn at any point bisects the angle between the lines joining the point to the foci whereas normal bisects the supplementary angle between the lines.

Equation of Pair of Tangents

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $SS_1 = T^2$

where, $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

and $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

Equation of Chord Bisected at a Given Point

If $P(x_1, y_1)$ bisects the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the equation of chord bisected at the point P is $T = S_1$

where, $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$ and $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$.

Equation of Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T = 0$,

where $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

Diameter

The equation of a diameter bisecting a system of parallel chord of slope m of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2m} x$.

Conjugate Diameter

Two diameters of a hyperbola are said to be conjugate diameters, if each bisects the chords parallel to the other.

Two lines $y = m_1x$ and $y = m_2x$ are conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $m_1m_2 = \frac{b^2}{a^2}$.

Director Circle

The equation of director circle of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 - b^2.$$

Pole and Polar

The polar of a point (x_1, y_1) with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $T = 0$.

Asymptotes

If the length of perpendicular drawn from a point on the hyperbola to a straight line tends to zero as the point moves to infinity, then the straight line is known as asymptotes.

The equations of two asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ are } y = \pm \frac{b}{a} x \text{ or } \frac{x}{a} \pm \frac{y}{b} = 0.$$

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Rectangular Hyperbola

A hyperbola whose asymptotes are perpendicular to each other is called rectangular hyperbola.

Or

If the length of transverse and conjugate axes of any hyperbola is equal, then it is called rectangular hyperbola.

Various Form of Rectangular Hyperbola

1. **Cartesian Form**

(i) The general equation of the rectangular hyperbola is $x^2 - y^2 = a^2$.

(ii) If we take coordinate axes along the asymptotes of a rectangular hyperbola, then the equation of rectangular hyperbola becomes $xy = c^2$, where c is any constant.

2. **Parametric Form**

The parametric equations of rectangular hyperbola $xy = c^2$ are $x = ct$ and $y = \frac{c}{t}$, where t is any parameter.

Properties of Rectangular Hyperbola $x^2 - y^2 = a^2$

(i) The equations of asymptotes of the rectangular hyperbola are $y = \pm x$.

(ii) The transverse and conjugate axes of a rectangular hyperbola are equal in length.

(iii) Eccentricity $(e) = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{2}$

Properties of Rectangular Hyperbola $xy = c^2$

(i) The equation of chord joining t_1 and t_2 is $x + yt_1t_2 - c(t_1 + t_2) = 0$.

(ii) The equation of tangent at (x_1, y_1) is $xy_1 + x_1y = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

(iii) The equation of tangent at $(ct, c/t)$ to the hyperbola is $x/t + yt = 2c$.

(iv) The point of intersection of tangents at $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

(v) The equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(vi) The equation of normal at t is $xt^3 - yt - ct^4 + c = 0$.

(vii) The equation of the chord of the hyperbola $xy = c^2$, whose middle point is (x_1, y_1), is $T = S_1$, i.e. $xy_1 + x_1y = 2x_1y_1$.

(viii) The slope of the tangent at the point $(ct, c/t)$ is $-1/t^2$, which is always negative. Hence, tangents drawn at any point to $xy = c^2$ would always make an obtuse angle with the X-axis.

(ix) The slope of the normal at the point $(ct, c/t)$ is t^2, which is always positive. Hence, normals drawn to $xy = c^2$ at any point would always make an acute angle with the X-axis.

Notice Board

JEE 2017 IMPORTANT DATES

S. No.	Description	Day, Date and Time
1.	JEE (Main) 2017 [Offline exam]	Sun, 02-04-2017
2.	Results of JEE (Main) 2017	Thu, 27-04-2017
3.	Registration for JEE (Advanced) 2017	Fri, 28-04-2017, 10:00 IST to Tue, 02-05-2017, 17:00 IST
4.	Registration with Late Fee	Wed, 03-05-2017 to Thu, 04-05-2017, 17:00 IST
5.	Admit card available for downloading	Wed, 10-05-2017, 10:00 IST to Sun, 21-05-2017, 09:00 IST
6.	JEE (Advanced) 2017 Paper-1 Paper-2	Sun, 21-05-2017 09:00-12:00 IST 14:00-17:00 IST
7.	(a) Online display of ORS and scanned responses (b) Request from candidates for review of scanned responses	Wed, 31-05-2017, 10:00 IST to Sat, 03-06-2017, 17:00 IST
8.	Online display of answer keys	Sun, 04-06-2017, 10:00 IST
9.	Feedback and comments on answer keys from the candidates	Sun, 04-06-2017, 10:00 IST to Tue, 06-06-2017, 17:00 IST
10.	Declaration of JEE (Advanced) results	Sun, 11-06-2017, 10:00 IST
11.	Online registration for Architecture Aptitude Test (AAT)	Sun, 11-06-2017, 10:00 IST to Mon, 12-06-2017, 17:00 IST
12.	Architecture Aptitude Test	Wed, 14-06-2017 09:00-12:00 IST
13.	Declaration of AAT results	Sun, 18-06-2017
14.	Seat Allotment (Tentative)	Mon, 19-06-2017 to Tue, 18-07-2017

*IST-Indian Standard Time

MASTER STROKES

1. The locus of point of intersection of tangents at the ends of normal chord of the hyperbola $x^2 - y^2 = a^2$, is

(a) $y^4 - x^4 = 4a^2x^2y^2$ (b) $y^2 - x^2 = 4a^2x^2y^2$
(c) $a^2(y^2 - x^2) = 4x^2y^2$ (d) $y^2 + x^2 = 4a^2x^2y^2$

2. Let A and B be two fixed points and P be another point in the plane, moves in such a way that $k_1PA + k_2PB = k_3$, where k_1, k_2 and k_3 are real constants. Then, the locus of P is a/an

(a) circle, if $k_1 = 0$ and $k_2, k_3 > 0$
(b) circle, if $k_1 > 0$, $k_2 < 0$ and $k_3 = 0$
(c) ellipse, if $k_1 = k_2 > 0$ and $k_3 > 0$
(d) hyperbola, if $k_2 = -1$ and $k_1, k_3 > 0$

3. The equation of the line passing through the centre of a rectangular hyperbola is $x - y - 1 = 0$. If one of its asymptotes is $3x - 4y - 6 = 0$, then the equation of the other asymptote is

(a) $4x + 3y + 17 = 0$ (b) $4x - 3y + 8 = 0$
(c) $3x - 2y + 15 = 0$ (d) None of these

4. If radii of director circles of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $2r$, r respectively and e_e, e_h are the eccentricities of the ellipse, hyperbola respectively, then

(a) $2e_h^2 - e_e^2 = 6$ (b) $e_e^2 - 4e_h^2 = 6$
(c) $4e_h^2 - e_e^2 = 6$ (d) None of these

5. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. The locus of the point which divides the line segment between these two points in the ratio 1 : 2, is

(a) $16x^2 + 10xy + y^2 = 2$ (b) $16x^2 - 10xy + y^2 = 2$
(c) $16x^2 + 10xy + y^2 = 4$ (d) None of these

6. If $P(a\sec\alpha, b\tan\alpha)$ and $Q(a\sec\beta, b\tan\beta)$ are two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that $\alpha - \beta = 2\theta$ (a constant), then PQ touches the hyperbola

(a) $\frac{x^2}{a^2\sec^2\theta} - \frac{y^2}{b^2} = 1$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2\sec^2\theta} = 1$
(c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cos^2\theta$ (d) None of these

7. The locus of the point of intersection of the tangents at the end points of normal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is

(a) $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 + b^2)^2$
(b) $\frac{a^6}{x^2} - \frac{b^6}{y^2} = (a^2 + b^2)^2$
(c) $\frac{a^6}{x^2} - \frac{b^6}{y^2} = (a^2 - b^2)^2$
(d) $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$

8. If the line $2x + \sqrt{6}\,y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is

(a) $(-2, \sqrt{6})$ (b) $(-5, 2\sqrt{6})$ (c) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (d) $(4, -\sqrt{6})$

9. If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then the eccentricity e of the hyperbola satisfies

(a) $1 < e < \frac{2}{\sqrt{3}}$ (b) $e = \frac{2}{\sqrt{3}}$ (c) $e = \frac{\sqrt{3}}{2}$ (d) $e > \frac{2}{\sqrt{3}}$

10. The normal at P to a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M, respectively. If locus of the mid-point of LM is a hyperbola, then eccentricity of the hyperbola is

(a) $\frac{e+1}{e-1}$ (b) $\frac{e}{\sqrt{e^2-1}}$
(c) e (d) None of these

11. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal to that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then the equation of ellipse is

(a) $x^2 + 2y^2 = 2$ with foci at $(\pm 1, 0)$
(b) $x^2 + 2y^2 = 2$ with foci at $(\pm \sqrt{2}, 0)$
(c) $x^2 + 2y^2 = 4$ with foci at $(\pm 1, 0)$
(d) $x^2 + 2y^2 = 4$ with foci at $(\pm \sqrt{2}, 0)$

12. If a variable straight line $x\cos\alpha + y\sin\alpha = p$ which is a chord of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(b > a)$ subtends a right angle at the centre of the hyperbola, then it always touches a fixed circle whose radius is

(a) $\frac{ab}{\sqrt{b-2a}}$ (b) $\frac{a}{\sqrt{a-b}}$ (c) $\frac{ab}{\sqrt{b^2-a^2}}$ (d) $\frac{ab}{\sqrt{b+a}}$

13. If $H(x, y) = 0$ represents the equation of a hyperbola and $A(x, y) = 0$, $C(x, y) = 0$ the joint equation of its asymptotes and the conjugate hyperbola respectively, then for any point (α, β) in the plane, $H(\alpha, \beta)$, $A(\alpha, \beta)$ and $C(\alpha, \beta)$ are in

(a) AP (b) GP
(c) HP (d) None of these

14. The point of intersection of two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the product of whose slopes is c^2, lies on the curve

(a) $y^2 - b^2 = c^2(x^2 + a^2)$
(b) $y^2 + a^2 = c^2(x^2 - b^2)$
(c) $y^2 + b^2 = c^2(x^2 - a^2)$
(d) $y^2 - a^2 = c^2(x^2 + b^2)$

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15. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant with change in α?
(a) Abscissae of vertices *(b)* Abscissae of foci
(c) Eccentricity *(d)* Directrix

16. If tangent to any member of family of hyperbolas $xy = 4\sin^2\theta, \theta \in (0, 2\pi) - \{\pi\}$, is not a normal to any member of family of circles $x^2 + y^2 - 2x - 2y + \mu = 0$, where μ is any real parameter, then θ belongs to
(a) $\left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$ *(b)* $\left(0, \frac{\pi}{6}\right)$ *(c)* $\left(\frac{11\pi}{6}, 2\pi\right)$ *(d)* All of these

17. The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B. The equation of a common tangent with positive slope to the circle as well as to the hyperbola, is
(a) $2x - \sqrt{5}y - 20 = 0$ *(b)* $2x - \sqrt{5}y + 4 = 0$
(c) $3x - 4y + 8 = 0$ *(d)* $4x - 3y + 4 = 0$

18. The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B. The equation of the circle with AB as its diameter, is
(a) $x^2 + y^2 - 12x + 24 = 0$ *(b)* $x^2 + y^2 + 12x + 24 = 0$
(c) $x^2 + y^2 + 24x - 12 = 0$ *(d)* $x^2 + y^2 - 24x - 12 = 0$

19. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the X-axis, then the eccentricity of the hyperbola is
(a) $\sqrt{2}$ *(b)* 2 *(c)* $\sqrt{3}$ *(d)* 1

20. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then the equation of the hyperbola is
(a) $\frac{x^2}{3} - \frac{y^2}{2} = 1$ *(b)* $x^2 - 3y^2 = 3$
(c) $\frac{x^2}{2} - \frac{y^2}{3} = 1$ *(d)* $3x^2 - y^2 = 3$

21. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then the focus of the hyperbola is at
(a) (2, 0) *(b)* (0, 2) *(c)* (3, 0) *(d)* (0, 3)

22. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the X-axis at (9, 0), then the eccentricity of the hyperbola is
(a) $\sqrt{\frac{5}{2}}$ *(b)* $\sqrt{\frac{3}{2}}$ *(c)* $\sqrt{2}$ *(d)* $\sqrt{3}$

23. The equation of the hyperbola whose foci are $(-2, 0)$, $(2, 0)$ and eccentricity 2, is given by
(a) $3x^2 - y^2 = 3$ *(b)* $-x^2 + 3y^2 = 3$
(c) $-3x^2 + y^2 = 3$ *(d)* $x^2 - 3y^2 = 3$

24. Suppose an ellipse and a hyperbola have the same pair of foci on the X-axis with centres at the origin and they intersect at (2, 2). If the eccentricity of the ellipse is $\frac{1}{2}$, then the eccentricity of the hyperbola is
(a) $\sqrt{\frac{7}{4}}$ *(b)* $\sqrt{\frac{7}{3}}$ *(c)* $\sqrt{\frac{5}{4}}$ *(d)* $\sqrt{\frac{5}{3}}$

25. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are
(a) $\left(\pm\frac{9}{2\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ *(b)* $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{9}{2\sqrt{2}}\right)$
(c) $(3\sqrt{3}, -2\sqrt{2})$ *(d)* $(-3\sqrt{3}, 2\sqrt{2})$

26. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ coincide with the foci of the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$, then b^2 is equal to
(a) 7 *(b)* 8 *(c)* 10 *(d)* 9

27. The angle between pair of tangents to the curve $7x^2 - 12y^2 = 84$ from the point $M(1, 2)$, is
(a) $2\tan^{-1}\frac{1}{2}$ *(b)* $2\tan^{-1}2$
(c) $2\left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right)$ *(d)* $2\tan^{-1}3$

28. Consider the chords of the parabola $y^2 = 4x$ which touches the hyperbola $x^2 - y^2 = 1$, the locus of the point of intersection of the tangents drawn to the parabola at the extremities of such chords is a conic section having latusrectum λ, the value of λ is
(a) 1 *(b)* 2 *(c)* 3 *(d)* 4

29. If there exist two points P and Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that $OP \perp OQ$, where O is the origin, then the number of points in the XY-plane from where pair of perpendicular tangents can be drawn to the hyperbola, is
(a) 0 *(b)* 1 *(c)* 2 *(d)* infinite

30. If four points are taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two and let $\alpha, \beta, \gamma, \delta$ be the inclinations to either asymptote of the straight lines joining these points to the centre, then $\tan\alpha\tan\beta\tan\gamma\tan\delta$ is equal to
(a) 1 *(b)* 2 *(c)* -2 *(d)* -1

Answers

1. (c)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (a)
6. (a)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (b)
11. (a)	**12.** (c)	**13.** (a)	**14.** (c)	**15.** (b)
16. (d)	**17.** (b)	**18.** (a)	**19.** (b)	**20.** (b)
21. (b)	**22.** (b)	**23.** (a)	**24.** (b)	**25.** (a)
26. (a)	**27.** (c)	**28.** (a)	**29.** (a)	**30.** (a)

TRIGONOMETRY

Trigonometry is a branch of Mathematics that studies relationships involving lengths and angles of triangles.

Measurement of Angles

There are three systems for measurement of an angle.

(i) Sexagesimal System

In this system, angle is measured in degree, minute and second. A complete rotation describes 360°.

1 right angle = 90° (90 degree)

1° = 60′ (60 minute)

1′ = 60″ (60 second)

(ii) Circular System

In this system, unit of measurement is in radian. One radian (written as 1^C) is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

(iii) Centesimal or French System

In this system, angle is measured in grade, minute and second.

1 right angle = 100^g (100 grade)

1 grade = 100′ (100 minute)

1 minute = 100″ (100 second)

Important Relation

Following are some important relations :

(i) π radian = 180°

(ii) $1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \approx 57^\circ 17' 45''$

(iii) $x \text{ radian} = x \cdot \left(\frac{180}{\pi}\right)^\circ$

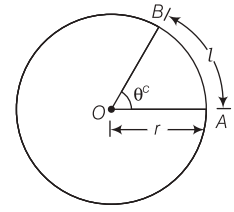
(iv) $x^\circ = x \cdot \left(\frac{\pi}{180}\right)$ radian

(v) If D is the number of degree, R is the number of radian and G is the number of grade in an angle θ, then

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

Relation between an Angle and an Arc

Let r be the radius of the circle, l be the length of segment and θ^c be the central angle. Then,



(i) $\theta = \frac{l}{r}$

(ii) Area of sector $OAB = \frac{1}{2} r^2 \theta$

POINTS TO REMEMBER

- Minute hand makes 6° angle (= π / 30 radian).
- Hour hand makes 30° angle (= π / 6 radian).
- The angle between two consecutive numbers in a clock is 30° (= π/ 6 radian).

Trigonometric Ratios

In ΔDEF, right angled at E, there are six possible ratios, each ratio is named as follows :

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{P}{H}$$

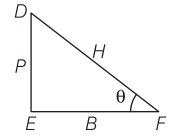
$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{B}{H}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{P}{B}$$

$$\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{H}{P}$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{H}{B}$$

$$\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{B}{P}$$



DO YOU Know?

A perfect number N is one in which the sum of its divisors equal N. The problem is that it has yet to be proven whether or not any odd perfect numbers exist.

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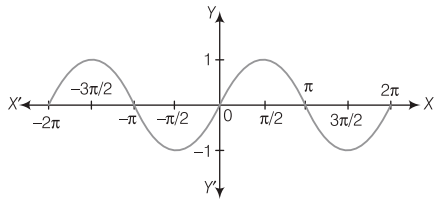
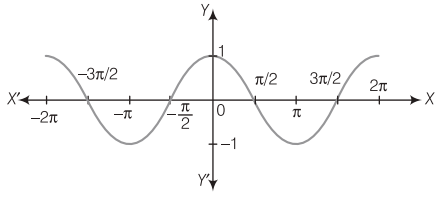
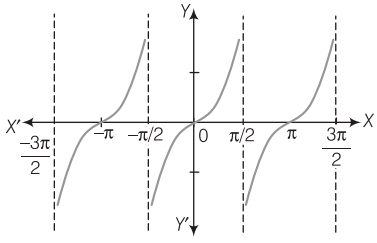
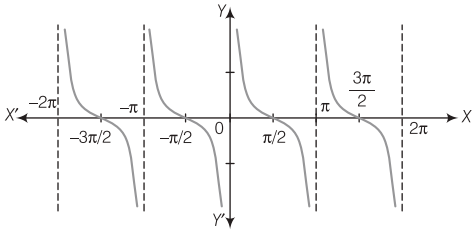
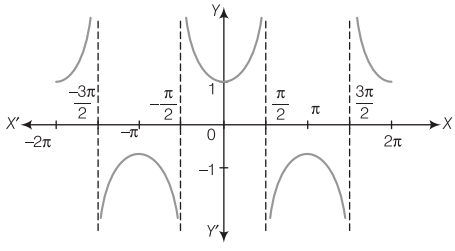
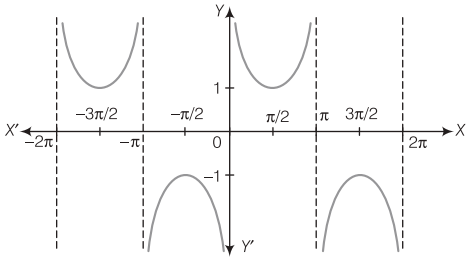
Trigonometric Functions

All six trigonometric functions with their domain, range, graph and other details are shown in the table given below :

Function	Domain	Range	Other Details	Graphical Representation
$f(x) = \sin x$	R	$[-1, 1]$	$\sin x = 0$, if $x = n\pi$, $n \in Z$ $\sin x > 0$, if $x \in (2n\pi, (2n+1)\pi)$, $n \in Z$ $\sin x < 0$, if $x \in ((2n-1)\pi, 2n\pi)$, $n \in Z$	
$f(x) = \cos x$	R	$[-1, 1]$	$\cos x = 0$, if $x = (2n+1)\frac{\pi}{2}$, $n \in Z$ $\cos x > 0$, if $x \in \left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right)$, $n \in Z$ $\cos x < 0$, if $x \in \left((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right)$, $n \in Z$	
$f(x) = \tan x$	$R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in Z$	R	$\tan x = 0$, if $x = n\pi$, $n \in Z$ $\tan x > 0$, if $x \in \left(n\pi, \left(n+\frac{1}{2}\right)\pi\right)$, $n \in Z$ $\tan x < 0$, if $x \in \left(\left(n-\frac{1}{2}\right)\pi, n\pi\right)$, $n \in Z$	
$f(x) = \cot x$	$R - \{n\pi\}, n \in Z$	R	$\cot x = 0$, if $x = (2n+1)\frac{\pi}{2}$, $n \in Z$ $\cot x > 0$, if $x \in \left(n\pi, \left(n+\frac{1}{2}\right)\pi\right)$, $n \in Z$ $\cot x < 0$, if $x \in \left(\left(n-\frac{1}{2}\right)\pi, n\pi\right)$, $n \in Z$	
$f(x) = \sec x$	$R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in Z$	$(-\infty, -1] \cup [1, \infty)$, $n \in Z$	$\sec x = 1$, when $\cos x = 1$ $\Rightarrow x = 2n\pi$, $n \in Z$ $\sec x = -1$, when $\cos x = -1$ $\Rightarrow x = (2n+1)\pi$, $n \in Z$	
$f(x) = \operatorname{cosec} x$	$R - \{n\pi\}$, $n \in Z$	$(-\infty, -1] \cup [1, \infty)$	$\operatorname{cosec} x = 1$, when $\sin x = 1$ $\Rightarrow x = (2n+1)\frac{\pi}{2}$, $n \in Z$ $\operatorname{cosec} x = -1$, when $\sin x = -1$ $\Rightarrow x = (2n-1)\frac{\pi}{2}$, $n \in Z$	

Trigonometry Identities

An equation is an identity, if it is true for all values of the variable or variables involved in it. Similarly, trigonometric identities are equalities involving trigonometric functions, which is true for all the values of the variable or variables occurring in it.

Basic Trigonometric Identities

A.	$\sin(-x) = -\sin x$	$\operatorname{cosec}(-x) = -\operatorname{cosec} x$
	$\cos(-x) = \cos x$	$\sec(-x) = \sec x$
	$\tan(-x) = -\tan x$	$\cot(-x) = -\cot x$
B.	$\sin\left(\frac{\pi}{2} \mp x\right) = \cos x$	$\sin(\pi \pm x) = \mp \sin x$
	$\cos\left(\frac{\pi}{2} \mp x\right) = \pm \sin x$	$\cos(\pi \pm x) = -\cos x$
	$\tan\left(\frac{\pi}{2} \mp x\right) = \pm \cot x$	$\tan(\pi \pm x) = \pm \tan x$

Pythagorean Identities

(i) $\sin^2 x + \cos^2 x = 1$
(ii) $1 + \tan^2 x = \sec^2 x$
(iii) $1 + \cot^2 x = \operatorname{cosec}^2 x$

Identities Based on Sum and Difference of Two Angles

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$
(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$
(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$
(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$
(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
(vi) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
(vii) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$
(viii) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$
(ix) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
(x) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

SPEED BOOSTER

- If $\cos x - \cos y = a$ and $\sin x - \sin y = b$, then $\sin(x + y) = \dfrac{-2ab}{a^2 + b^2}$, $\cos(x + y) = \dfrac{b^2 - a^2}{a^2 + b^2}$ and $\tan(x + y) = \dfrac{2ab}{a^2 - b^2}$.
- If $\cos x + \cos y = a$ and $\sin x + \sin y = b$, then $\sin(x + y) = \dfrac{2ab}{a^2 + b^2}$, $\cos(x + y) = \dfrac{a^2 - b^2}{a^2 + b^2}$ and $\tan(x + y) = \dfrac{2ab}{a^2 - b^2}$.

Identities Based on Sum of Three or More Angles

(i) $\sin(A + B + C) = \cos A \cos B \cos C$
$(\tan A + \tan B + \tan C - \tan A \cdot \tan B \cdot \tan C)$
$= \sin A \cos B \cos C + \cos A \sin B \cos C$
$+ \cos A \cos B \sin C - \sin A \sin B \sin C$

(ii) $\cos(A + B + C) = \cos A \cos B \cos C$
$(1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$
$= \cos A \cos B \cos C - \sin A \sin B \cos C$
$- \cos A \sin B \sin C - \sin A \cos B \sin C$

(iii) $\tan(A + B + C)$
$= \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(iv) $\cot(A + B + C)$
$= \dfrac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot A \cot B + \cot B \cot C + \cot C \cot A - 1}$

(v) $\sin(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n$
$(S_1 - S_3 + S_5 - S_7 + \ldots)$

(vi) $\cos(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n$
$(1 - S_2 + S_4 - S_6 + \ldots)$

(vii) $\tan(A_1 + A_2 + \ldots + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$

where, $S_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n$
$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \ldots$
= sum of tangents taken two at a time.
$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots$
= sum of tangents taken three at a time and so on.

Note If $A_1 = A_2 = \ldots = A_n = A$, then we have $S_1 = n \tan A$, $S_2 = {}^nC_2 \tan^2 A$, $S_3 = {}^nC_3 \tan^3 A, \ldots$

Identities Based on Multiple of an Angle

(i) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(ii) $\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1$
$= \cos^2 A - \sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(iii) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

(iv) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(v) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(vi) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

SPEED BOOSTER

- $\sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta)=\frac{1}{4}\sin 3\theta$
- $\cos\theta\cos(60^\circ-\theta)\cos(60^\circ+\theta)=\frac{1}{4}\cos 3\theta$
- $\tan\theta\tan(60^\circ-\theta)\tan(60^\circ+\theta)=\tan 3\theta$
- $\cos A\cos 2A\cos 4A\cos 8A\ldots\cos 2^{n-1}A=\frac{1}{2^n\sin A}\sin(2^n A)$
- $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\{\alpha+(n-1)\beta\}$
$$=\frac{\sin\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$
- $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}$
$$=\frac{\cos\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Identities Based on Transformation of Product into Sum and Difference

(i) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$
(ii) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$
(iii) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$
(iv) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Identities Based on Transformation of Sum and Difference into Products

(i) $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
(ii) $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
(iii) $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
(iv) $\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
(v) $\tan A\pm\tan B=\frac{\sin(A\pm B)}{\cos A\cdot\cos B}$
(vi) $\cot A\pm\cot B=\frac{\sin(B\pm A)}{\sin A\cdot\sin B}$

Conditional Identities

When the angles A, B and C satisfy a relation like $A+B+C=\pi$, many interesting trigonometric identities can be established.

1. If $A+B+C=\pi$, then
(i) $\sin(B+C)=\sin A$
(ii) $\cos B=-\cos(C+A)$
(iii) $\cos(A+B)=-\cos C$
(iv) $\sin C=\sin(A+B)$
(v) $\tan(C+A)=-\tan B$
(vi) $\cot A=-\cot(B+C)$

2. If $A+B+C=\pi$, then
(i) $\cos\left(\frac{A+B}{2}\right)=\sin\frac{C}{2}$
(ii) $\cos\frac{C}{2}=\sin\left(\frac{A+B}{2}\right)$
(iii) $\sin\left(\frac{C+A}{2}\right)=\cos\frac{B}{2}$
(iv) $\sin\left(\frac{A}{2}\right)=\cos\left(\frac{B+C}{2}\right)$
(v) $\tan\left(\frac{B+C}{2}\right)=\cot\left(\frac{A}{2}\right)$
(vi) $\tan\left(\frac{B}{2}\right)=\cot\left(\frac{C+A}{2}\right)$

3. If $A+B+C=\pi$, then
(i) $\sin^2A+\sin^2B-\sin^2C=2\sin A\sin B\cos C$
(ii) $\cos^2A+\cos^2B+\cos^2C=1-2\cos A\cos B\cos C$
(iii) $\sin^2A+\sin^2B+\sin^2C=1-2\sin A\sin B\cos C$

4. If $A+B+C=\pi$, then
(i) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
(ii) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
(iii) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
(iv) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

5. If $A+B+C=\pi$, then
(i) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$
(ii) $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$
(iii) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$
(iv) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

6. If $A+B+C=\pi$, then
(i) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$
(ii) $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$
(iii) $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)=4\sin A\sin B\sin C$
(iv) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$
(v) $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$

7. If $A+B+C=\pi$, then
(i) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
(ii) $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
(iii) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
(iv) $\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
(v) $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin C\sin A}+\frac{\cos C}{\sin A\sin B}=2$

Trigonometric Ratios of Some Special Angles

	7.5°	15°	18°	22.5°	36°	67.5°	75°
sin	$\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{8+2\sqrt{6}+2\sqrt{2}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$	$2+\sqrt{3}$	$\sqrt{5+2\sqrt{5}}$	$\sqrt{2}+1$	$\sqrt{1+\frac{2}{\sqrt{5}}}$	$\sqrt{2}-1$	$2-\sqrt{3}$

SPEED BOOSTER

- Maximum and minimum values of some standard trigonometric functions are given below :

	Maximum value	Minimum value
$f(x) = a\cos x \pm b\sin x,\ x \in R$	$\sqrt{a^2+b^2}$	$-\sqrt{a^2+b^2}$
$f(x) = a\cos x \pm b\sin x + C,\ x \in R$	$C+\sqrt{a^2+b^2}$	$C-\sqrt{a^2+b^2}$

- If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 + \ldots + \sin\theta_n = n$, then $\sin\theta_1 = \sin\theta_2 = \ldots = \sin\theta_n = 1$
- If $\cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_n = n$, then $\cos\theta_1 = \cos\theta_2 = \ldots = \cos\theta_n = 1$

MASTER STROKES

1. For $0 < x < \frac{\pi}{6}$, all the values of $\tan^2 3x\cos^2 x - 4\tan 3x\sin 2x + 16\sin^2 x$ lie in the interval

(a) $\left(0, \frac{121}{36}\right)$ (b) $\left(1, \frac{121}{9}\right)$

(c) $(-1, 0)$ (d) None of these

2. The least value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$ is

(a) $4+\sqrt{10}$ (b) $4-\sqrt{10}$

(c) 0 (d) None of these

3. If $x + y + z = \pi$, $\tan x \tan z = 2$ and $\tan y \tan z = 18$, then $\tan^2 z$ is equal to

(a) 15 (b) 16 (c) 19 (d) 20

4. Biggest among $(\sin 1 + \cos 1)$, $(\sqrt{\sin 1} + \sqrt{\cos 1})$, $\sqrt{1+\sin 2}$ and 1 is

(a) $\sin 1 + \cos 1$ (b) $\sqrt{\sin 1} + \sqrt{\cos 1}$

(c) $\sqrt{1+\sin 2}$ (d) 1

5. The expression $2^{\sin\theta} + 2^{-\cos\theta}$ is minimum, when θ is equal to

(a) $2n\pi + \frac{\pi}{4}, n \in Z$ (b) $2n\pi + \frac{7\pi}{4}, n \in Z$

(c) $n\pi \pm \frac{\pi}{4}, n \in Z$ (d) None of these

6. In an acute angled ΔABC, $\tan A \tan B \tan C$

(a) ≥ 3 (b) $\geq \sqrt{3}$

(c) $\geq 3\sqrt{3}$ (d) None of these

7. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, then the difference between the maximum and minimum values of u^2 is given by

(a) $(a-b)^2$ (b) $2\sqrt{a^2+b^2}$

(c) $(a+b)^2$ (d) $2(a^2+b^2)$

8. If both θ and ϕ are acute angles and $\sin\theta = \frac{1}{2}$, $\cos\phi = \frac{1}{3}$, then the value of $\theta + \phi$ belongs to

(a) $(\pi/3, \pi/2]$ (b) $(\pi/2, 2\pi/3]$

(c) $(2\pi/3, 5\pi/6]$ (d) $(5\pi/6, \pi]$

9. If $a = \frac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta}$ is equal to

(a) $\frac{1}{a}$ (b) a (c) $1-a$ (d) $1+a$

10. If A and B are positive acute angles satisfying the equations $3\cos^2 A + 2\cos^2 B = 4$ and $\frac{3\sin A}{\sin B} = \frac{2\cos B}{\cos A}$, then $A + 2B$ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

11. In ΔABC, if $\cos 3A + \cos 3B + \cos 3C = 1$, then one angle must be exactly equal to
(a) 30° (b) 60° (c) 90° (d) 120°

12. If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is equal to
(a) $\frac{1-\sqrt{7}}{4}$ (b) $\frac{4-\sqrt{7}}{3}$ (c) $-\frac{4+\sqrt{7}}{3}$ (d) $\frac{\sqrt{7}+1}{4}$

13. If $\alpha, \beta, \gamma, \delta$ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2}$ is equal to
(a) $2\sqrt{1-k}$ (b) $2\sqrt{1+k}$ (c) $2\sqrt{k}$ (d) $2\sqrt{k+2}$

14. Let α, β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -\frac{21}{65}$ and $\cos\alpha + \cos\beta = -\frac{27}{65}$, then the value of $\cos\frac{\alpha-\beta}{2}$ is
(a) $-\frac{6}{65}$ (b) $\frac{3}{\sqrt{130}}$ (c) $\frac{6}{65}$ (d) $-\frac{3}{\sqrt{130}}$

15. If ABC is an acute angled triangle, then the minimum value of $\tan^4 A + \tan^4 B + \tan^4 C$ will be
(a) 729 (b) 27 (c) $81\sqrt{3}$ (d) None of these

16. Two parallel chords are drawn on the same side of the centre of a circle of radius 12 units. It is found that they subtend 72° and 144° angles at the centre of the circle. The perpendicular distance (in units) between the chords is
(a) 6 (b) 4 (c) 3 (d) 2

17. If $0 < x < \frac{\pi}{2}$ and $\sin^n x + \cos^n x \geq 1$, then n belongs to the interval
(a) $[2, \infty)$ (b) $(-\infty, 2]$ (c) $[-1, 1]$ (d) $(-\infty, 1]$

18. The number of integral triplets (a, b, c) such that $a + b\cos 2x + c\sin^2 x = 0$ for all x, is
(a) 0 (b) 1 (c) 3 (d) infinite

19. Let $\frac{3\pi}{4} < \theta < \pi$ and $\sqrt{2\cot\theta + \frac{1}{\sin^2\theta}} = k - \cot\theta$, then k is equal to
(a) 1 (b) -1 (c) 0 (d) $\frac{1}{2}$

20. Let $f: (-1, 1) \to R$ be such that $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$ for $\theta \to \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then, the values of $f\left(\frac{1}{3}\right)$ are
(a) $1 \pm \frac{\sqrt{3}}{2}$ (b) $1 \pm \sqrt{\frac{2}{3}}$ (c) $1 \pm \sqrt{\frac{1}{3}}$ (d) $1 \pm \sqrt{\frac{1}{2}}$

21. Suppose θ and ϕ $(\neq 0)$ are such that $\sec(\theta + \phi)$, $\sec\theta$ and $\sec(\theta - \phi)$ are in AP. If $\cos\theta = k\cos\left(\frac{\phi}{2}\right)$ for some k, then k is equal to
(a) ± 1 (b) ± 2 (c) $\pm\sqrt{2}$ (d) $\pm\frac{1}{\sqrt{2}}$

22. The value of $\sec 40° + \sec 80° + \sec 160°$ will be
(a) 4 (b) -4 (c) 6 (d) 8

23. The expression $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A}$ can be written as
(a) $\sin A\cos A + 1$ (b) $\sec A\ \text{cosec}\ A + 1$
(c) $\tan A + \cot A$ (d) $\sec A + \text{cosec}\ A$

24. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R$ and $k \geq 1$. Then, $f_4(x) - f_6(x)$ equals
(a) $\frac{1}{4}$ (b) $\frac{1}{12}$ (c) $\frac{1}{6}$ (d) $\frac{1}{3}$

25. The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$, is
(a) 2 (b) 3 (c) 4 (d) 6

26. Let $\theta, \phi \in [0, 2\pi]$ be such that $2\cos\theta(1 - \sin\phi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1$, $\tan(2\pi - \theta) > 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$. Then, ϕ satisfies
(a) $\theta < \phi < \frac{\pi}{2}$ (b) $\frac{\pi}{2} < \phi < \frac{4\pi}{3}$
(c) $\frac{4\pi}{3} < \phi < \frac{3\pi}{2}$ (d) $\frac{3\pi}{2} < \phi < 2\pi$

27. The positive integral value of n (> 3) satisfying $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$, is
(a) 6 (b) 7 (c) 8 (d) 4

28. If $\theta = \frac{2\pi}{2009}$, then $\cos\theta\cos 2\theta\cos 3\theta \ldots \cos 1004\theta$ equals
(a) 0 (b) $\frac{1}{2^{2008}}$ (c) $\frac{1}{2^{1004}}$ (d) $-\frac{1}{2^{1004}}$

29. If $\sin A\sin B\sin C + \cos A\cos B = 1$, then the value of $\sin C$ is
(a) 1 (b) 1/2 (c) 0 (d) -1

30. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ apart. If the chords subtend at the centre, angles $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where $k > 0$, then the value of $[k]$ is
(a) 1 (b) 2 (c) 3 (d) 4

Answers

1. (a)	2. (b)	3. (b)	4. (b)	5. (b)
6. (c)	7. (d)	8. (b)	9. (b)	10. (b)
11. (d)	12. (c)	13. (b)	14. (d)	15. (b)
16. (a)	17. (b)	18. (d)	19. (b)	20. (a)
21. (c)	22. (c)	23. (b)	24. (b)	25. (a)
26. (b)	27. (b)	28. (c)	29. (a)	30. (b)

PROPERTIES OF TRIANGLES AND CIRCLES CONNECTED WITH THEM

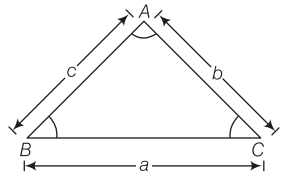
In any ΔABC, the side BC, opposite to $\angle A$ is denoted by a; the sides CA and AB, opposite to $\angle B$ and $\angle C$ are denoted by b and c, respectively. Semi-perimeter of the triangle is denoted by s and its area by Δ.

Sine Rule

It states that in any triangle, "The sides of a triangle are proportional to the sine of opposite angles."

In ΔABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Note Above equality can be written as $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, to find angles.

Cosine Rule

If the length of the three sides of triangle are known, then we can find all the angles by using cosine rule which states that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

and $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

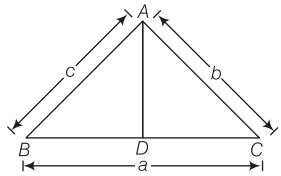
Projection Formulae

In any ΔABC,

$$a = b\cos C + c\cos B$$
$$b = c\cos A + a\cos C$$
$$c = a\cos B + b\cos A$$

i.e. any side of a triangle is equal to the sum of the projections of other two sides on it.

Note In ΔABC, projection of AB on BC = $BD = c\cos B$ and projection of AC on BC = $CD = b\cos C$

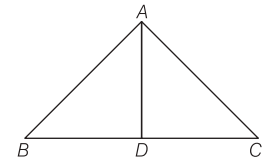


REMEMBER

- **Apollonius Theorem**

The sum of the squares of any two sides of a triangle equals twice the square of half the third side, along with twice the square of the median bisecting the third side. Thus, in ΔABC, we have

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$



- **Napier's Analogy** (Tangent's rule)

For any ΔABC, $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$,

$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$, $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

- **Mollweide's Formula**

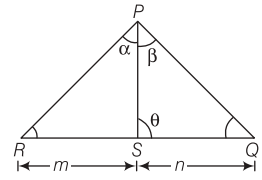
For any ΔABC, $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$, $\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}$.

- ***m-n* Theorem**

Let S be a point on the side RQ of ΔPQR such that $RS : SQ = m : n$, $\angle PSQ = \theta$, $\angle RPS = \alpha$ and $\angle QPS = \beta$. Then,

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m+n)\cot\theta = n\cot R - m\cot Q$



Half Angle Formulae

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ (iv) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(v) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (vi) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(vii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (viii) $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

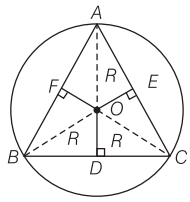
(ix) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

where, s = Semi-perimeter of the triangle $= \frac{a+b+c}{2}$

Circumcircle

(i) The circle which passes through the vertices of a triangle, is called circumcircle.

(ii) The centre of circumcircle is the point of intersection of perpendicular bisectors of the sides of the triangle.



REMEMBER

- $\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=R$
- $\frac{2\sin A}{a}=\frac{2\sin B}{b}=\frac{2\sin C}{c}=\frac{1}{R}$
- $R=\frac{abc}{4\Delta}$

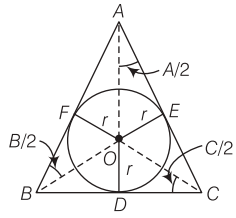
SPEED BOOSTER

- Circumcentre of an obtuse angled triangle lies outside the triangle.
- Circumcentre of a right angled triangle is the mid-point of hypotenuse.
- Circumcentre of an acute angled triangle lies inside the triangle.

Incircle

(i) Circle which can be inscribed in a triangle, touching its each side, is called incircle.

(ii) The centre of incircle is the point of intersection of bisectors of the angles of the triangle.



REMEMBER

- $r=\frac{\Delta}{s}$
- $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$
- $\cos A+\cos B+\cos C=1+\frac{r}{R}$
- $r=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}=\frac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}}=\frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

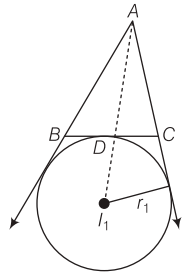
SPEED BOOSTER

- If p_1, p_2 and p_3 are respectively the lengths of perpendiculars drawn from the vertices to opposite sides of a triangle, then
$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}$$
and
$$p_1p_2p_3=\frac{(abc)^2}{8R^3}$$
- If x, y and z are the lengths of perpendiculars drawn from the circumcentre to the sides a, b and c of ΔABC respectively, then
$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$$

Escribed Circle

(i) The circle which touches the side BC and two sides AB and AC produced of ΔABC is called escribed circle opposite to $\angle A$ and its radius is denoted by r_1. Similarly, r_2 and r_3 denote radii of escribed circles opposite to $\angle B$ and $\angle C$.

(ii) The centres of escribed circles are called the excentres.



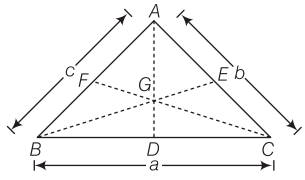
REMEMBER

- $r_1=\frac{\Delta}{s-a}, r_2=\frac{\Delta}{s-b}, r_3=\frac{\Delta}{s-c}$
- $r_1=s\tan\frac{A}{2}, r_2=s\tan\frac{B}{2}, r_3=s\tan\frac{C}{2}$
- $r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$; $r_2=4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$, $r_3=4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$ and $r_1=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$; $r_2=\frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$; $r_3=\frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$
- **Distance between excentres**
$$I_1I_2=4R\cos\frac{C}{2},\quad I_2I_3=4R\cos\frac{A}{2};\ I_3I_1=4R\cos\frac{B}{2}$$
- $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$; $r_1+r_2+r_3-r=4R$
- $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2Rr}$
- $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$

Centroid (G)

(i) The point of intersection of medians of a triangle is called centroid of the triangle.

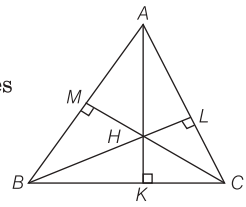
(ii) Centroid divides the median in the ratio 2 : 1. Therefore,
$AG=\frac{2}{3}AD, DG=\frac{1}{3}AD$



(iii) Length of the medians of ΔABC are
$$AD=\frac{1}{2}\sqrt{2b^2+2c^2-a^2},$$
$$CF=\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$$
and
$$BE=\frac{1}{2}\sqrt{2a^2+2c^2-b^2}$$

Orthocentre (*H*)

The point of intersection of altitudes of a triangle is called orthocentre.



REMEMBER

- Centroid (G) of a triangle is situated on the line joining its circumcentre (O) and orthocentre (H) and divides this line in the ratio 1 : 2, i.e. $OG : GH = 1 : 2$.
- Orthocentre of an acute angled triangle lies inside the triangle.
- Orthocentre of an obtuse angled triangle lies outside the triangle.
- Orthocentre of a right angled triangle lies at the vertex which contain right angle.
- The distances of orthocentre from the vertices and the sides of a triangle are given below :

From vertices	From sides
$AH = 2R\cos A$	$DH = 2R\cos B\cos C$
$BH = 2R\cos B$	$EH = 2R\cos A\cos C$
$CH = 2R\cos C$	$FH = 2R\cos A\cos B$

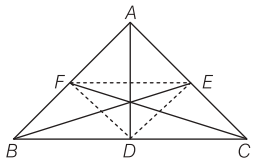
- If x, y and z are the distances of the vertices of ΔABC respectively from the orthocentre, then $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$.

SPEED BOOSTER

- The distance between circumcentre and incentre is $\sqrt{R^2-2Rr}$.
- The distance between circumcentre and excentre is $R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$.
- The distance between circumcentre and orthocentre is $R\sqrt{1-8\cos A\cos B\cos C}$.
- The distance between the incentre and excentre is given by $I_1 = 4R\sin\left(\frac{A}{2}\right)$; $I_2 = 4R\sin\left(\frac{B}{2}\right)$ and $I_3 = 4R\sin\left(\frac{C}{2}\right)$.

Pedal Triangle

The triangle formed by joining the end points of the altitudes drawn from the vertices of the triangle is called pedal triangle.



(i) ΔDEF is the pedal triangle.

(ii) Measures of sides and angles of pedal triangle are given below :

Angles	Sides
$\angle FDE = 180° - 2A$	$EF = a\cos A$
$\angle DEF = 180° - 2B$	$DF = b\cos B$
$\angle EFD = 180° - 2C$	$DE = c\cos C$

(iii) Area $= \frac{R^2}{2}\sin 2A\sin 2B\sin 2C$

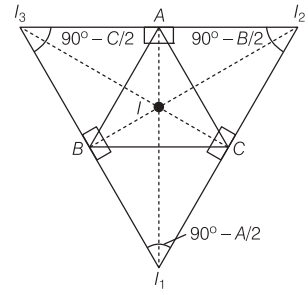
(iv) Circumradius, $R' = \frac{R}{2}$

(v) Inradius, $r = 2R\cos A\cos B\cos C$

Note For an obtuse angled triangle, measures of angles are $2A$, $2B$ and $2C - 180°$ and sides are $a\cos A$, $b\cos B$, $-c\cos C$.

Excentral Triangle

Let ABC be a triangle, I be the incentre and I_1, I_2, I_3 be the centres of the escribed circles opposite to A, B, C respectively, then $I_1 I_2 I_3$ is called excentral triangle of ΔABC, whose sides and angles are given in the following table :

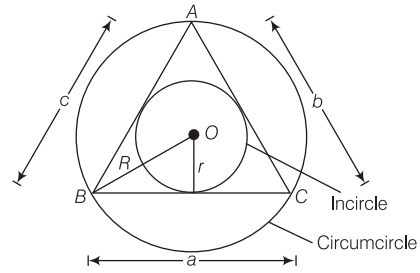


Angles	Sides
$\angle I_3I_1I_2 = 90° - \frac{A}{2}$	$I_1I_2 = 4R\cos\left(\frac{C}{2}\right)$
$\angle I_1I_2I_3 = 90° - \frac{B}{2}$	$I_2I_3 = 4R\cos\left(\frac{A}{2}\right)$
$\angle I_2I_3I_1 = 90° - \frac{C}{2}$	$I_1I_3 = 4R\cos\left(\frac{B}{2}\right)$

REMEMBER

- Circumradius (R') of excentral triangle is $2R$, where R is the circumradius of ΔABC.
- Area of excentral triangle $= 8R^2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

Area of a Triangle



where, r = inradius , R = circumradius

1. When Two Sides and the Included Angle are Given

Δ = Area of $\Delta ABC = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

$= \frac{1}{2}\times$ (Product of two sides) $\times$ sine of included angle

2. When Three Sides of a Triangle are Given (Heron's Formula)

Δ = Area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$,

where a, b and c are the sides and $s = \dfrac{a+b+c}{2}$

3. When Circumradius and Three Sides are Given

Δ = Area of $\Delta ABC = \dfrac{abc}{4R}$

4. When Two Angles and Included Side are Given

$$\Delta = \text{Area of } \Delta ABC = \frac{a^2}{2} \times \frac{\sin B \sin C}{\sin (B+C)}$$

$$= \frac{b^2}{2} \times \frac{\sin A \sin C}{\sin (A+C)} = \frac{c^2}{2} \times \frac{\sin B \sin A}{\sin (B+A)}$$

5. When Inradius (r) and Semi-perimeter (s) of a Triangle are Given

Δ = Area of $\Delta ABC = r \cdot s$

6. When Triangle is Right Angled Triangle

$$\Delta = \text{Area of } \Delta ABC = \frac{1}{2} \times \text{base} \times \text{height}$$

7. When the Lengths of Three Medians are Given

$$\Delta = \text{Area of } \Delta ABC = \frac{4}{3}\sqrt{s(s-M_1)(s-M_2)(s-M_3)}$$

where, $s = \dfrac{M_1 + M_2 + M_3}{2}$; M_1, M_2 and M_3 are the lengths of medians.

8. When the Lengths of Altitudes are Given

$$\Delta = \text{Area of } \Delta ABC = \frac{1}{4\sqrt{A\left(A-\frac{1}{A_1}\right)\left(A-\frac{1}{A_2}\right)\left(A-\frac{1}{A_3}\right)}}$$

where, $A = \dfrac{\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3}}{2}$; A_1, A_2 and A_3 are the lengths of altitudes.

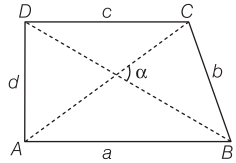
Note $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{2\Delta}{abc}$

Area of a Quadrilateral

Area of quadrilateral $ABCD$

$= \dfrac{1}{2}$ (Product of diagonals) $\times$ (sine of included angle)

$= \dfrac{1}{2} \times BD \times AC \times \sin\alpha$



or area of quadrilateral $ABCD$ can also be obtained by

$$\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$$

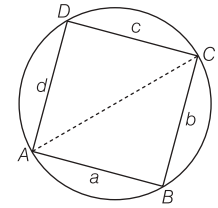
where, $\alpha = \dfrac{\angle A + \angle C}{2}$ and $s = \dfrac{a+b+c+d}{2}$.

Area of a Cyclic Quadrilateral

Area of cyclic quadrilateral $= \dfrac{1}{2}(ab+cd)\sin B$

Here, $\angle B$ is given by

$$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab+cd)}$$



or area of cyclic quadrilateral can also be obtained by $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $s = \dfrac{a+b+c+d}{2}$.

SPEED BOOSTER

- Sum of the opposite angles of a cyclic quadrilateral is 180°.
- In cyclic quadrilateral, sum of the product of opposite sides is equal to the product of diagonals. Thus, in cyclic quadrilateral $ABCD$, we have
  $AC \cdot BD = AB \cdot CD + BC \cdot AD$
- Circumradius of cyclic quadrilateral,
  $R = \dfrac{1}{4\Delta}\sqrt{(ac+bd)(ad+bc)(ab+cd)}$

Regular Polygons

If the polygon has n sides, then we have

(i) Sum of the interior angles $= (n-2)\pi$

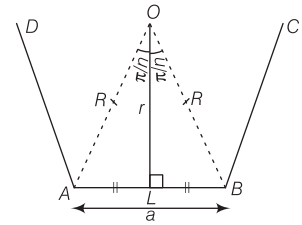
(ii) Each interior angle $= \dfrac{(n-2)\pi}{n}$

(iii) Radius of circumscribed circle, $R = \dfrac{a}{2}\operatorname{cosec}\left(\dfrac{\pi}{n}\right)$, where a is the length of each side of regular polygon.

(iv) Radius of inscribed circle, $r = \dfrac{a}{2}\cdot\cot\left(\dfrac{\pi}{n}\right)$, where a is the length of each side of regular polygon.

Area of Regular Polygon

If the polygon has n sides, then



Required area $= n \times$ Area of ΔOAB

$= \dfrac{1}{4}na^2\cot\left(\dfrac{\pi}{n}\right)$ (in terms of side a)

$= nr^2\tan\left(\dfrac{\pi}{n}\right)$ (in terms of inradius r)

$= \dfrac{n}{2}R^2\sin\left(\dfrac{2\pi}{n}\right)$ (in terms of circumradius R)

MASTER STROKES

1. If the area (Δ) and an angle (θ) of a triangle are given, when the side opposite to the given angle is minimum, then the lengths of the remaining two sides are

(a) $\sqrt{\frac{2\Delta}{\sin\theta}}, \sqrt{\frac{3\Delta}{\sin\theta}}$ (b) $\sqrt{\frac{2\Delta}{\sin\theta}}, \sqrt{\frac{2\Delta}{\sin\theta}}$

(c) $\sqrt{\frac{4\Delta}{\sin\theta}}, \sqrt{\frac{4\Delta}{\sin\theta}}$ (d) $\sqrt{\frac{6\Delta}{\sin\theta}}, \sqrt{\frac{6\Delta}{\sin\theta}}$

2. If the sides of a triangle are in AP and the greatest angle of the triangle exceeds the least by 90°, then sine of the third angle is

(a) $\frac{\sqrt{5}}{4}$ (b) $\frac{\sqrt{6}}{4}$

(c) $\frac{\sqrt{7}}{4}$ (d) None of these

3. If p, q and r are the lengths of the internal bisectors of angles A, B and C respectively of ΔABC, then $\frac{1}{p}\cos\frac{A}{2} + \frac{1}{q}\cos\frac{B}{2} + \frac{1}{r}\cos\frac{C}{2}$ is equal to

(a) $\frac{1}{a} + \frac{1}{b} - \frac{1}{c}$ (b) $\frac{1}{a} + \frac{1}{c} - \frac{1}{b}$

(c) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ (d) $\frac{1}{b} + \frac{1}{c} - \frac{1}{a}$

4. In ΔABC, if the altitudes are in HP, then

(a) angles, A, B and C are in AP

(b) sides a, b and c are in AP

(c) $\sin A, \sin B$ and $\sin C$ are in AP

(d) None of the above

5. In ΔABC, least value of $\frac{e^A}{A} + \frac{e^B}{B} + \frac{e^C}{C}$ is equal to

(a) $\frac{9}{\pi}e^{\pi/3}$ (b) $\frac{\pi}{3}e^{\pi/3}$

(c) $\frac{\pi}{9}e^{\pi/3}$ (d) None of these

6. In ΔABC, x, y and z are the distances of incentre from angular points A, B and C, respectively. If $\frac{xyz}{abc} = \frac{\lambda r}{s}$, then λ is equal to

(a) 1 (b) 2

(c) 3 (d) None of these

7. If Δ denotes the area of any triangle with semi-perimeter, then

(a) $\Delta < \frac{s^2}{2}$ (b) $\Delta > \frac{s^2}{4}$

(c) $\Delta < \frac{s^2}{4}$ (d) $\Delta < s^2$

8. In ΔABC, $AB = 2$, $BC = 4$, $CA = 3$. If D is the mid-point of BC, then the correct statement is

(a) $\cos B \neq \frac{11}{16}$ (b) $\cos C \neq \frac{7}{8}$

(c) $AD \neq 2.4$ (d) $AD^2 = 2.5$

9. In ΔABC, if $a^2 + b^2 + c^2 = ac + \sqrt{3}ab$, then the triangle is

(a) equilateral

(b) right angled and isosceles

(c) right angled but not isosceles

(d) None of the above

10. In ΔABC, AD and BE are the medians drawn through the angular points A and B, respectively. If $\angle DAB = 2 \angle ABE = 36°$ and $AD = 6$ units, then circumradius of the triangle is equal to

(a) $(3 - \sqrt{5})$ cosec C (b) $(3 + \sqrt{5})$ cosec C

(c) $2(3 - \sqrt{5})$ cosec C (d) $2(3 + \sqrt{5})$ cosec C

11. If the median AM, angle bisector AD and altitude AH drawn from vertex A of ΔABC divide $\angle A$ into four equal parts (D lying between H and M), then

(a) $A = \frac{\pi}{3}$ (b) $A = \frac{\pi}{2}$

(c) $\frac{AC}{AB} = \sqrt{2} + 1$ (d) $\frac{AC}{AB} = \frac{1}{\sqrt{2} + 1}$

12. In ΔABC, medians AD and BE are drawn. If $AD = 4$, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$, then the area of ΔABC is

(a) $\frac{64}{3\sqrt{3}}$ (b) $\frac{8}{3\sqrt{3}}$ (c) $\frac{16}{3\sqrt{3}}$ (d) $\frac{32}{3\sqrt{3}}$

13. In ΔABC, if $r_1 = 36$, $r_2 = 18$ and $r_3 = 12$, then the area of triangle is

(a) 216 sq units (b) 316 sq units

(c) 326 sq units (d) None of these

14. In ΔABC, if $r_1 = 36$, $r_2 = 18$ and $r_3 = 12$, then the perimeter of triangle is

(a) 36 units (b) 18 units

(c) 72 units (d) None of these

15. If D is the mid-point of side BC of ΔABC and AD is perpendicular to AC, then

(a) $b^2 = a^2 - c^2$ (b) $a^2 + b^2 = 5c^2$

(c) $3b^2 = a^2 - c^2$ (d) $3a^2 = b^2 - 3c^2$

16. In ΔABC, if $r_1 = 2r_2 = 3r_3$, then $a : b : c$ equals

(a) 3 : 4 : 5 (b) 5 : 3 : 4

(c) 5 : 4 : 3 (d) None of these

17. In ΔABC, if $r_1 = 8$, $r_2 = 12$ and $r_3 = 24$, then the side a is equal to

(a) 16 (b) 20

(c) 12 (d) None of these

18. If the sides of a triangle are $\sin\alpha$, $\cos\alpha$ and $\sqrt{1 + \sin\alpha\cos\alpha}$ for some $0 < \alpha < \frac{\pi}{2}$. Then, the greatest angle of the triangle is

(a) 150° (b) 90° (c) 120° (d) 60°

19. In ΔABC, if $\frac{1}{b+c}+\frac{1}{c+a}=\frac{3}{a+b+c}$, then $\angle C$ equals

(a) $90°$ (b) $60°$
(c) $45°$ (d) $30°$

20. If the sides a, b and c of ΔABC are in AP and $\cos\theta_1=\frac{a}{b+c}$, $\cos\theta_2=\frac{b}{a+c}$ and $\cos\theta_3=\frac{c}{a+b}$, then $\tan^2\frac{\theta_1}{2}+\tan^2\frac{\theta_3}{2}$ equals

(a) $2/3$ (b) 1
(c) $\sqrt{5}/3$ (d) None of these

21. In ΔABC, let a, b and c denote the lengths of the sides opposite to vertices A, B and C, respectively. Suppose $a=6$, $b=10$ and the area of triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and r denotes the radius of the incircle of the triangle, then r^2 is equal to

(a) 2 (b) 4 (c) 3 (d) 6

22. If the angles A, B and C of a triangle are in AP and a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$ is

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\sqrt{3}$

23. Let ABC be a triangle such that $\angle ACB=\frac{\pi}{6}$ and a, b and c denote the lengths of the sides opposite to A, B and C, respectively. Then, the value of x for which $a=x^2+x+1$, $b=x^2-1$ and $c=2x+1$, is

(a) $-(2+\sqrt{3})$ (b) $1+\sqrt{3}$ (c) $2+\sqrt{3}$ (d) $4\sqrt{3}$

24. For a regular polygon, let r and R be the radii of the inscribed and circumscribed circles. A false statement among the following is

(a) there is a regular polygon with $\frac{r}{R}=\frac{2}{3}$
(b) there is a regular polygon with $\frac{r}{R}=\frac{\sqrt{3}}{2}$
(c) there is a regular polygon with $\frac{r}{R}=\frac{1}{2}$
(d) there is a regular polygon with $\frac{r}{R}=\frac{1}{\sqrt{2}}$

25. Let PQR be a triangle of area Δ with $a=2$, $b=\frac{7}{2}$ and $c=\frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles P, Q and R, respectively. Then, $\frac{2\sin P-\sin 2P}{2\sin P+\sin P}$ equals

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$ (c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$

26. Let $ABCD$ is a trapezium such that AB and CD are parallel and $BC\perp CD$. If $\angle ADB=\theta$, $BC=p$ and $CD=q$, then AB is equal to

(a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (b) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$
(c) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

27. In ΔPQR, P is the largest angle and $\cos P=\frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then, possible lengths of the sides of the triangle are

(a) 16, 18 (b) 18, 22 (c) 22, 24 (d) 16, 20

28. In a triangle, the sum of two sides is x and the product of the same two sides is y. If $x^2-c^2=y$, where c is the third side of the triangle, then the ratio of the inradius to the circumradius of the triangle is

(a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$
(c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$

29. In ΔABC, if $\frac{\sin A}{c\sin B}+\frac{\sin B}{c}+\frac{\sin C}{b}=\frac{c}{ab}+\frac{b}{ac}+\frac{a}{bc}$, then the value of A is

(a) $120°$ (b) $90°$ (c) $60°$ (d) $30°$

30. In ΔABC, if $2b=a+c$ and $A-C=90°$, then $\sin B$ equals

(a) $\frac{\sqrt{7}}{5}$ (b) $\frac{\sqrt{5}}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{5}}{3}$

Answers

1. (b)	2. (c)	3. (c)	4. (c)	5. (a)
6. (a)	7. (c)	8. (d)	9. (c)	10. (b)
11. (b)	12. (d)	13. (a)	14. (c)	15. (c)
16. (c)	17. (c)	18. (c)	19. (b)	20. (a)
21. (b)	22. (d)	23. (b)	24. (a)	25. (c)
26. (a)	27. (b)	28. (b)	29. (b)	30. (c)

MIND Shakers

The day before yesterday, I was 25 and the next year, I will be 28. This is true only once a day in a year.
What day is my birthday?

SOLUTIONS OF TRIANGLE

In a triangle, there are six elements *viz*. three sides and three angles. In plane geometry, we have learned that if three of the elements are given, atleast one of which must be a side, then the other three elements can be uniquely determined. The procedure of determining unknown elements from the known elements is called solving a triangle.

Solution of a Right Angled Triangle

Following are two cases with respect to solution of a right angled triangle :

***Case* I** *When two sides are given*

Let the triangle be right angled at C. Then, we can determine the remaining elements as given in the following table

	Given	Required
(i)	a, b	$\tan A = \frac{a}{b}, B = 90^\circ - A, c = \frac{a}{\sin A}$
(ii)	a, c	$\sin A = \frac{a}{c}, b = c \cos A, B = 90^\circ - A$

***Case* II** *When a side and an acute angle are given*

Let ABC be a right triangle and right angled at C.

In this case, we can determine the remaining elements as given in the following table

	Given	Required
(i)	a, A	$B = 90^\circ - A, b = a \cot A, c = \frac{a}{\sin A}$
(ii)	c, A	$B = 90^\circ - A, a = c \sin A, b = c \cos A$

POINTS TO REMEMBER

- In any right angled triangle, the orthocentre coincides with the vertex containing the right angled.
- The mid-point of the hypotenuse of a right angled triangle is equidistant from the three vertices of the triangle.
- The mid-point of the hypotenuse of a right angled triangle is the circumcentre of the triangle.

General Solution of a Triangle

Following cases arise while solving a triangle :

***Case* I** *When three sides a, b and c are given*

In this case, the remaining elements are determined by using the following formulae

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$

$\sin A = \frac{2\Delta}{bc}$, $\sin B = \frac{2\Delta}{ac}$, $\sin C = \frac{2\Delta}{ab}$ $\tan \frac{A}{2} = \frac{\Delta}{s(s-a)}$,

$\tan \frac{B}{2} = \frac{\Delta}{s(s-b)}$, $\tan \frac{C}{2} = \frac{\Delta}{s(s-c)}$ $A + B + C = 180^\circ$

***Case* II** *When two sides a, b and the included angle C are given*

In this case, we use the following formulae

$\Delta = \frac{1}{2} ab \sin C$, $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$, $\frac{A+B}{2} = 90^\circ - \frac{C}{2}$,

$c = \frac{a \sin C}{\sin A}$

***Case* III** *When one side a and two angles A and B are given*

In this case, we use the following formulae to determine the remaining elements

$$A + B + C = 180^\circ \Rightarrow C = 180^\circ - A - B$$

$$b = \frac{a \sin B}{\sin A} \text{ and } c = \frac{a \sin C}{\sin A}, \Delta = \frac{1}{2} ca \sin B$$

***Case* IV** *When two sides a, b and the $\angle A$ opposite to one side are given*

In this case, we use the following formulae

$$\sin B = \frac{b}{a} \sin A \qquad \text{...(i)}$$

$$C = 180^\circ - (A + B) \text{ and } c = \frac{a \sin C}{\sin A}$$

From Eq. (i), the following possibilities will arise :

(a) *When A is an acute angle and $a < b \sin A$*

In this case, the relation $\sin B = \frac{b}{a} \sin A$ gives that $\sin B > 1$, which is impossible.

Hence, no triangle is possible.

(b) *When A is an acute angle and $a = b \sin A$*

In this case, only one triangle is possible which is right angled at B.

(c) *When A is an acute angle and $a > b \sin A$*

In this case, there are two values of B given by $\sin B = \frac{b \sin A}{a}$, say B_1 and B_2 such that $B_1 + B_2 = 180^\circ$.

Side c can be obtained by using $c = \frac{a \sin C}{\sin A}$

MASTER STROKES

1. In ΔABC, if $B = 90°$, then $\tan^2 \frac{A}{2}$ equals

(a) $\frac{a-b}{a+b}$ (b) $\frac{b-c}{b+c}$ (c) $\frac{c-a}{c+a}$ (d) $\frac{b+c}{b-c}$

2. If two sides and included angle of a triangle are respectively $3+\sqrt{3}$, $3-\sqrt{3}$ and $60°$, then the third side is

(a) $2\sqrt{2}$ (b) $4\sqrt{2}$ (c) $3\sqrt{2}$ (d) $5\sqrt{2}$

3. If the angles of a triangle are $30°$ and $45°$ and the included side is $(\sqrt{3}+1)$ cm, then the area of the triangle is

(a) $\frac{\sqrt{3}-1}{2}$ cm^2 (b) $\frac{\sqrt{3}+1}{2}$ cm^2

(c) $(\sqrt{3}-1)$ cm^2 (d) None of these

4. In ΔABC, if $b = 3$, $c = 4$ and $B = 60°$, then the number of triangles that can be formed is

(a) 0 (b) 1 (c) 2 (d) ∞

5. In ΔABC, if a, c and A are given and b_1, b_2 are two values of third side b such that $b_2 = 2b_1$, then $\sin A$ equals

(a) $\sqrt{\frac{9a^2-c^2}{8a^2}}$ (b) $\sqrt{\frac{9a^2-c^2}{8c^2}}$

(c) $\sqrt{\frac{9a^2+c^2}{8a^2}}$ (d) None of these

6. In ΔABC, if the length of the sides are $\sqrt{2}$, $\sqrt{6}$ and $\sqrt{8}$, then the measures of the angles are

(a) 30°, 60°, 90° (b) 45°, 75°, 60°

(c) 45°, 30°, 105° (d) None of these

7. In ΔABC, if $a = 100$, $c = 100\sqrt{2}$ and $A = 30°$, then B equals

(a) 45° or 135° (b) 105° or 15°

(c) 60° or 120° (d) None of these

8. In ΔABC, if $A = 45°$ and $B = 75°$, then $a + \sqrt{2}\,c$ equals

(a) b (b) $2b$ (c) $3b$ (d) $4b$

9. In ΔABC, if $a = 40$, $c = 40\sqrt{3}$ and $B = 30°$, then the triangle is

(a) isosceles (b) equilateral

(c) right angled (d) None of these

10. In ΔABC, if $a = \sqrt{3}+1$, $B = 30°$ and $C = 45°$, then c equals

(a) 3 (b) 2 (c) 4 (d) 5

11. In ΔABC, if $a = 4$, $b = 3$ and $\angle A = 60°$, then c is a root of the equation

(a) $c^2 - 3c - 7 = 0$ (b) $c^2 + 3c + 7 = 0$

(c) $c^2 - 3c + 7 = 0$ (d) $c^2 + 3c - 7 = 0$

12. In ΔABC, the lengths of the two larger sides are 10 and 9 units, respectively. If the angles are in AP, then the length of the third side can be

(a) $5 \pm \sqrt{6}$ (b) $3\sqrt{3}$

(c) 5 (d) 3

13. In ΔABC, if $b = \sqrt{3}+1$, $c = \sqrt{3}-1$ and $A = 60°$, then the value of $\tan\left(\frac{B-C}{2}\right)$ is

(a) -1 (b) $\frac{1}{2}$

(c) 1 (d) $-\frac{1}{2}$

14. If the angles A, B and C of ΔABC are in AP and $b : c = \sqrt{3} : \sqrt{2}$, then $\angle A$ is

(a) 75° (b) 60°

(c) 45° (d) 30°

15. In ΔABC, if $A = 120°$, $b = 2$ and $c = 30°$, then a equals

(a) $2\sqrt{3}$ (b) 2

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

16. In ΔABC, if $A = 30°$, $C = 105°$ and $b = 3\sqrt{2}$, then a equals

(a) $\sqrt{2}$ (b) 3 (c) $3\sqrt{2}$ (d) 2

17. In ΔABC, if $a = 3$, $b = 2\sqrt{3}$ and $c = \sqrt{3}$, then A equals

(a) 30° (b) 45° (c) 60° (d) 75°

18. In ΔABC, if $a = 5$, $B = 45°$ and $c = 2\sqrt{2}$, then b equals

(a) $\sqrt{2}$ (b) 6

(c) $2\sqrt{13}$ (d) $\sqrt{13}$

19. In ΔABC, if $\frac{a}{b^2-c^2} + \frac{c}{b^2-a^2} = 0$, then B equals

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

20. In the ambiguous case, if a, b and A are given and c_1, c_2 are the two values of the third side, then $(c_1 - c_2)^2 + (c_1 + c_2)^2 \tan^2 A$ is equal to

(a) 4 (b) $4a^2$

(c) $4b^2$ (d) $4c^2$

Answers

1. (b)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (b)
6. (a)	**7.** (b)	**8.** (b)	**9.** (a)	**10.** (b)
11. (a)	**12.** (a)	**13.** (c)	**14.** (a)	**15.** (a)
16. (b)	**17.** (c)	**18.** (d)	**19.** (c)	**20.** (b)

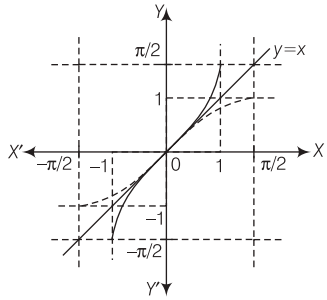
INVERSE TRIGONOMETRIC FUNCTION

We know that, inverse of a function exist iff it is one-one and onto. Now, as all trigonometric functions are many-one in their actual domain, therefore for inverse of these trigonometric functions to be exist, actual domain must be restricted, e.g. sine function can be restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, etc., and thus it becomes one-one and onto.

DOMAIN, RANGE AND GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

All six inverse trigonometric functions with their domain, range, graph and other details are given below.

1. $f(x) = \sin^{-1} x$



(i) Domain $(x) = [-1, 1]$

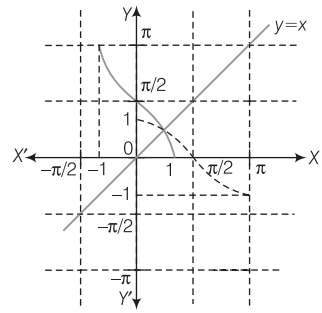
(ii) Range (principal value branch) $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) Increasing function

(iv) $x_1 > x_2 \Rightarrow \sin^{-1} x_1 > \sin^{-1} x_2$; $x_1, x_2 \in [-1, 1]$

(v) $\sin^{-1} x_1 > \sin^{-1} x_2 \Rightarrow x_1 > x_2$; $x_1, x_2 \in [-1, 1]$

2. $f(x) = \cos^{-1} x$



(i) Domain $(x) = [-1, 1]$

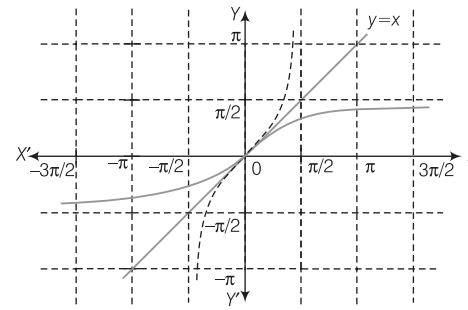
(ii) Range (principal value branch) $= [0, \pi]$

(iii) Decreasing function

(iv) $x_1 > x_2 \Rightarrow \cos^{-1} x_1 < \cos^{-1} x_2$; $x_1, x_2 \in [-1, 1]$

(v) $\cos^{-1} x_1 > \cos^{-1} x_2 \Rightarrow x_1 < x_2$; $x_1, x_2 \in [-1, 1]$

3. $f(x) = \tan^{-1} x$



(i) Domain $(x) = R$

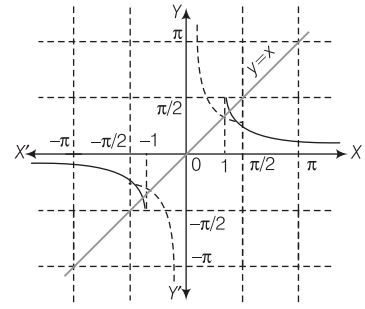
(ii) Range (principal value branch) $= \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(iii) Increasing function

(iv) $x_1 > x_2 \Rightarrow \tan^{-1} x_1 > \tan^{-1} x_2$; $x_1, x_2 \in R$

(v) $\tan^{-1} x_1 > \tan^{-1} x_2 \Rightarrow x_1 > x_2$; $x_1, x_2 \in R$

4. $f(x) = \text{cosec}^{-1} x$



(i) Domain $(x) = (-\infty, -1] \cup [1, \infty)$

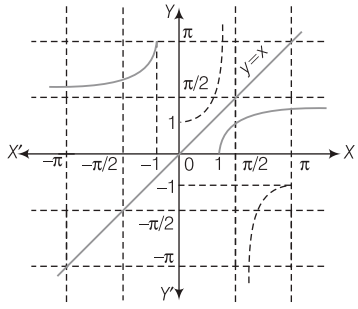
(ii) Range (principal value branch) $= \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(iii) Decreasing function

(iv) $x_1 > x_2 \Rightarrow \text{cosec}^{-1} x_1 < \text{cosec}^{-1} x_2$; $x_1, x_2 \in (-\infty, -1] \cup [1, \infty)$

(v) $\text{cosec}^{-1} x_1 > \text{cosec}^{-1} x_2 \Rightarrow x_1 < x_2$; $x_1, x_2 \in (-\infty, -1] \cup [1, \infty)$

5. $f(x) = \sec^{-1} x$



(i) Domain $(x) = R - (-1, 1)$

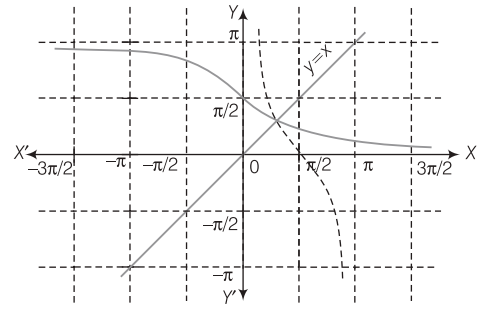
(ii) Range (principal value branch) $= [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(iii) Increasing function

(iv) $x_1 > x_2 \Rightarrow \sec^{-1} x_1 > \sec^{-1} x_2$, $x_1 x_2 \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1} x_1 > \sec^{-1} x_2 \Rightarrow x_1 > x_2$; $x_1, x_2 \in (-\infty, -1] \cup [1, \infty)$

6. $f(x) = \cot^{-1} x$



(i) Domain $(x) = R$

(ii) Range (principal value branch) $= (0, \pi)$

(iii) Decreasing function

(iv) $x_1 > x_2 \Rightarrow \cot^{-1} x_1 < \cot^{-1} x_2$; $x_1, x_2 \in R$

(v) $\cot^{-1} x_1 > \cot^{-1} x_2 \Rightarrow x_1 < x_2$; $x_1, x_2 \in R$

Note If any point (x_1, y_1) lies on the curve $y = f(x)$, then corresponding to it (y_1, x_1) lies on $y = f^{-1}(x)$. Since, points (x_1, y_1) and (y_1, x_1) are symmetrical about the line $y = x$, therefore the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about the line $y = x$.

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

There are following properties of inverse trigonometric functions.

Property I

If T is trigonometric function and $T^{-1}(x)$ is its inverse, then $T\{T^{-1}(x)\} = x$, $\forall\ x \in$ Domain of T^{-1}.

(i) $y = \sin(\sin^{-1} x) = x$; $x \in [-1, 1]$

(ii) $y = \cos(\cos^{-1} x) = x$; $x \in [-1, 1]$

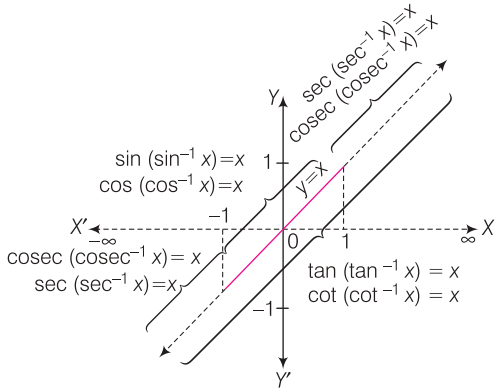
(iii) $y = \tan(\tan^{-1} x) = x$; $x \in (-\infty, \infty)$

(iv) $y = \text{cosec}(\text{cosec}^{-1} x) = x$; $x \in (-\infty, -1] \cup [1, \infty)$

(v) $y = \sec(\sec^{-1} x) = x$; $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $y = \cot(\cot^{-1} x) = x$; $x \in (-\infty, \infty)$

Graphically



Property II

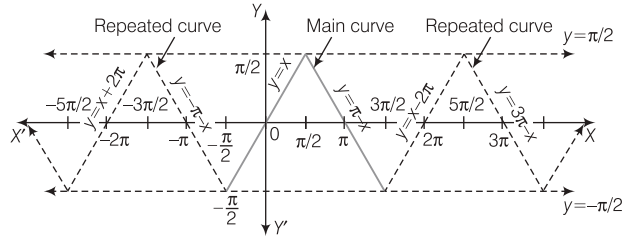
If T is trigonometric function and $T^{-1}(x)$ is its inverse, then $T^{-1}(T(x)) = x$, $\forall\ x \in D_T$, where D_T is principal value branch of inverse trigonometric function. If $x \notin D_T$, then $T^{-1}(T(x)) \neq x$ and in this case, graphical approach will help you to get the possible solution.

(i) Graph of $y = \sin^{-1}(\sin x)$

As we know that, $fog(x)$ is periodic with period of $g(x)$, so $T^{-1}(T(x))$ is periodic with period of $T(x)$. Hence, $y = \sin^{-1}(\sin x)$ is periodic with period 2π. Also, for length of interval 2π, we have

$$\sin^{-1}(\sin x) = \begin{cases} x; & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x; & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \end{cases}$$

Now, the graph of $y = \sin^{-1}(\sin x)$ can be plotted as



From the graph, it is clear that

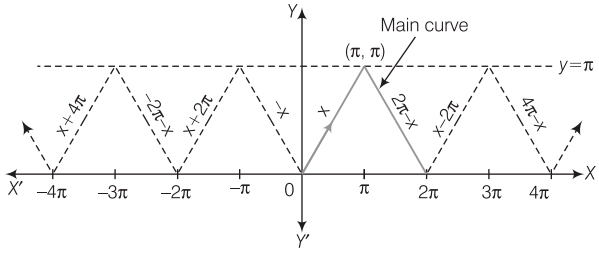
$$y = \sin^{-1}(\sin x) = \begin{cases} x + 2\pi\,; & -\frac{5\pi}{2} \le x \le \frac{-3\pi}{2} \\ -\pi - x\,; & -\frac{3\pi}{2} \le x \le \frac{-\pi}{2} \\ x\,; & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x\,; & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi\,; & \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \end{cases}$$

(ii) Graph of $y = \cos^{-1}(\cos x)$

Clearly, $y = \cos^{-1}(\cos x)$ is periodic with period 2π and

$$\cos^{-1}(\cos x) = \begin{cases} x\,; & 0 \le x \le \pi \\ 2\pi - x\,; & \pi \le x \le 2\pi \end{cases}$$

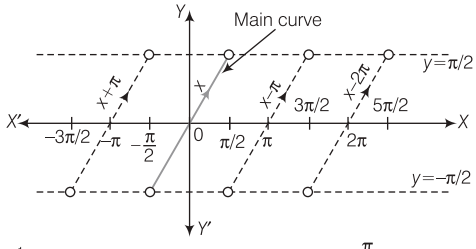
So, its graph can be plotted as



(iii) Graph of $y = \tan^{-1}(\tan x)$

Clearly, $y = \tan^{-1}(\tan x)$ is periodic with period π and $\tan^{-1}(\tan x) = x; \frac{-\pi}{2} < x < \frac{\pi}{2}$

Now, the graph of $y = \tan^{-1}(\tan x)$ can be plotted as



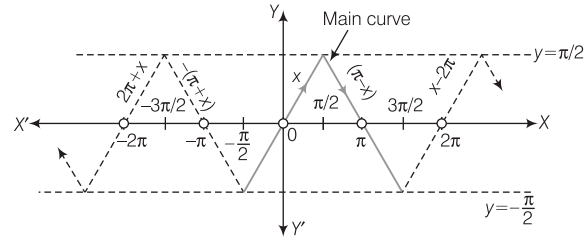
Note $\tan^{-1}(\tan x)$ is not defined for $x = (2n + 1)\frac{\pi}{2}, n \in Z$.

(iv) Graph of $y = \text{cosec}^{-1}(\text{cosec } x)$

Clearly, $y = \text{cosec}^{-1}(\text{cosec } x)$ is periodic with period 2π and

$$\text{cosec}^{-1}(\text{cosec } x) = \begin{cases} x\,; & x \in \left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \\ \pi - x\,; & x \in \left[\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right] \end{cases}$$

Now, the graph of $y = \text{cosec}^{-1}(\text{cosec } x)$ can be plotted as



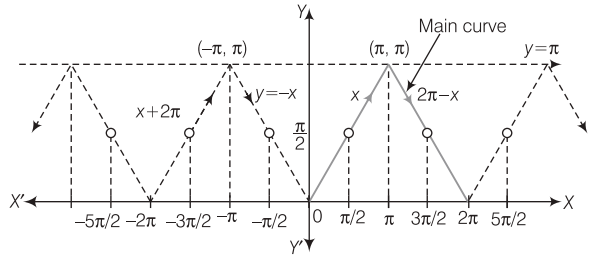
Note The graph of $\text{cosec}^{-1}(\text{cosec } x)$ is similar to $\sin^{-1}(\sin x)$ but exclude the points $x = n\pi, n \in Z$.

(v) Graph of $y = \sec^{-1}(\sec x)$

Clearly, $y = \sec^{-1}(\sec x)$ is periodic with period 2π

$$\text{and } \sec^{-1}(\sec x) = \begin{cases} x; & 0 \le x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < x \le \pi \\ 2\pi - x; & \pi \le x < \frac{3\pi}{2} \text{ or } \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

Now, the graph of $y = \sec^{-1}(\sec x)$ can be plotted as

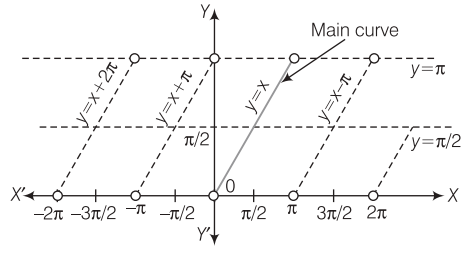


Note The graph of $\sec^{-1}(\sec x)$ is similar to $\cos^{-1}(\cos x)$ but exclude the points $x = (2n + 1)\frac{\pi}{2}, n \in Z$.

(vi) Graph of $y = \cot^{-1}(\cot x)$

Clearly, $y = \cot^{-1}(\cot x)$ is periodic with period π and $\cot^{-1}(\cot x) = x; 0 < x < \pi$.

Now, the graph of $y = \cot^{-1}(\cot x)$ can be plotted as



Note $\cot^{-1}(\cot x)$ is not defined when $x = n\pi, n \in Z$.

CHALLENGE Yourself

Rectify the following equality

$101 - 102 = 1$ by moving just one digit.

Property III

(i) $\sin^{-1}(-x) = -\sin^{-1} x;\ x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x;\ x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x;\ x \in (-\infty, \infty)$

(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x;\ x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x;\ x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x;\ x \in R$

Property IV

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x,\ x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x,\ x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, \text{ if } x > 0 \\ -\pi + \cot^{-1} x, \text{ if } x < 0 \end{cases}$

Property V

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},\ \forall\ x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ \forall\ x \in R$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2},\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

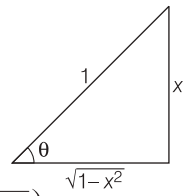
Property VI

(i) If $x > 0$, then we have

$$\sin^{-1} x = \cos^{-1}\sqrt{1 - x^2}$$

$$= \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right)$$

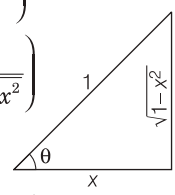
$$= \sec^{-1}\left(\frac{1}{\sqrt{1 - x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \theta$$



(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1 - x^2} = \tan^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right)$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1 - x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$$

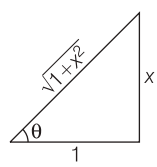
$$= \sec^{-1}\left(\frac{1}{x}\right) = \theta$$



(iii) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1 + x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1 + x^2}}\right)$

$$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(\sqrt{1 + x^2})$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{1 + x^2}}{x}\right) = \theta$$



Property VII

Sum and Difference Formulae

(i) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x + y}{1 - xy}\right), \text{ if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x + y}{1 - xy}\right), \\ \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x + y}{1 - xy}\right), \\ \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$

(ii) $\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x - y}{1 + xy}\right), \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\frac{x - y}{1 + xy}\right), \\ \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x - y}{1 + xy}\right), \\ \text{if } x < 0,\ y > 0 \text{ and } xy < -1 \end{cases}$

(iii) $\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1}(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}), \\ \text{if} -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ \text{or if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}), \\ \text{if } 0 < x, y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}), \\ \text{if} -1 \le x,\ y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$

(iv) $\sin^{-1} x - \sin^{-1} y = \begin{cases} \sin^{-1}(x\sqrt{1 - y^2} - y\sqrt{1 - x^2}), \\ \text{if} -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ \text{or if } xy > 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}(x\sqrt{1 - y^2} - y\sqrt{1 - x^2}), \\ \text{if } 0 < x \le 1, -1 \le y \le 0 \\ \text{and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}(x\sqrt{1 - y^2} - y\sqrt{1 - x^2}), \\ \text{if} -1 \le x < 0,\ 0 < y \le 1 \\ \text{and } x^2 + y^2 > 1 \end{cases}$

(v) $\cos^{-1} x + \cos^{-1} y = \begin{cases} \cos^{-1}(xy - \sqrt{1 - x^2}\sqrt{1 - y^2}), \\ \text{if} -1 \le x, y \le 1 \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}(xy - \sqrt{1 - x^2}\sqrt{1 - y^2}), \\ \text{if} -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$

(vi) $\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}(xy+\sqrt{1-x^2}\sqrt{1-y^2}),\\ \text{if}-1\le x,y\le 1 \text{ and } x\le y\\ -\cos^{-1}(xy+\sqrt{1-x^2}\sqrt{1-y^2}),\\ \text{if}-1\le y\le 0,\ 0<x\le 1 \text{ and } x\ge y\end{cases}$

Property VIII

(i) $2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}),\ \text{if}-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}),\ \text{if } x>\frac{1}{\sqrt{2}}\\ -\pi-\sin^{-1}(2x\sqrt{1-x^2}),\ \text{if } x<-\frac{1}{\sqrt{2}}\end{cases}$

(ii) $3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3),\ \text{if}-\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3),\ \text{if } x>\frac{1}{2}\\ -\pi-\sin^{-1}(3x-4x^3),\ \text{if } x<-\frac{1}{2}\end{cases}$

(iii) $2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1),\ \text{if } 0\le x\le 1\\ 2\pi-\cos^{-1}(2x^2-1),\ \text{if}-1\le x\le 0\end{cases}$

(iv) $3\cos^{-1}x=\begin{cases}\cos^{-1}(4x^3-3x),\ \text{if } \frac{1}{2}\le x\le 1\\ 2\pi-\cos^{-1}(4x^3-3x),\ \text{if}-\frac{1}{2}\le x\le\frac{1}{2}\\ 2\pi+\cos^{-1}(4x^3-3x),\ \text{if}-1\le x\le-\frac{1}{2}\end{cases}$

(v) $2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right),\ \text{if}-1<x<1\\ \pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right),\ \text{if } x>1\\ -\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right),\ \text{if } x<-1\end{cases}$

(vi) $2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\frac{2x}{1+x^2}\right),\ \text{if}-1\le x\le 1\\ \pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right),\ \text{if } x>1\\ -\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right),\ \text{if } x<-1\end{cases}$

(vii) $2\tan^{-1}x=\begin{cases}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ \text{if } 0\le x<\infty\\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ \text{if}-\infty<x\le 0\end{cases}$

(viii) $3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ \text{if}-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ \text{if } x>\frac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ \text{if } x<-\frac{1}{\sqrt{3}}\end{cases}$

SPEED BOOSTER

- If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2}$, then $x^2+y^2+z^2+2xyz=1$.
- If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$.
- If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then $xy+yz+zx=3$.
- If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then $x^2+y^2+z^2+2xyz=1$.
- If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then $xy+yz+zx=3$.
- If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$. In general,

  $\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{s_1-s_3+s_5-\ldots}{1-s_2+s_4-s_6+\ldots}\right)$

  where $s_1=\Sigma x_1, s_2=\Sigma x_1x_2, s_3=\Sigma x_1x_2x_3,\ldots$
- If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then $xy+yz+zx=1$.
- If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $x+y+z=xyz$.

REMEMBER

Inverse Hyperbolic Function

If $\sinh y=x$, then y is called the inverse hyperbolic sine of x and it is written as $y=\sinh^{-1}x$. Similarly, $\cosh^{-1}x$, $\tanh^{-1}x$, etc., can be defined. Let us look at some important results

$\sinh^{-1}x=\log_e(x+\sqrt{x^2+1}),\ x\in R$

$\cosh^{-1}x=\log_e(x+\sqrt{x^2-1}),\ x\ge 1$

$\tanh^{-1}x=\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right),\ |x|<1$

$\coth^{-1}x=\frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right),\ |x|>1$

$\text{sech}^{-1}x=\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ x\in(0,1]$

$\text{cosech}^{-1}x=\begin{cases}\log_e\left(\frac{1+\sqrt{1+x^2}}{x}\right),\ x>0\\ \log_e\left(\frac{1-\sqrt{1+x^2}}{x}\right),\ x<0\end{cases}$

FIND THE MISSING NUMBER.

3 5 9 15 23 33 ? 59

MASTER STROKES

1. If $\tan^{-1}\dfrac{x+1}{x-1}+\tan^{-1}\dfrac{x-1}{x}=\tan^{-1}(-7)+\pi$, then x equals

(a) 2 (b) 3 (c) 4 (d) None of these

2. Exhaustive set of values of parameter a so that $\sin^{-1}x-\tan^{-1}x=a$ has a solution, is

(a) $\left[-\dfrac{\pi}{6},\dfrac{\pi}{6}\right]$ (b) $\left[-\dfrac{\pi}{4},\dfrac{\pi}{4}\right]$

(c) $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ (d) None of these

3. If $\left|\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)\right|<\dfrac{\pi}{3}$, then x belongs to the interval

(a) $\left[-\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}\right]$ (b) $\left(-\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}\right)$

(c) $\left(0,\dfrac{1}{\sqrt{3}}\right)$ (d) None of these

4. If $\cos^{-1}(\cos 4)>3x^2-4x$, then

(a) $x\in\left(-\infty,\dfrac{2-\sqrt{6\pi-8}}{3}\right)$

(b) $x\in\left(\dfrac{2+\sqrt{6\pi-8}}{3},\infty\right)$

(c) $x\in\left(\dfrac{2-\sqrt{6\pi-8}}{3},\dfrac{2+\sqrt{6\pi-8}}{3}\right)$

(d) ∞

5. If $-1<x<0$, then $\sin\left\{\tan^{-1}\dfrac{1-x^2}{2x}-\cos^{-1}\dfrac{1-x^2}{1+x^2}\right\}$ is equal to

(a) 1 (b) -1

(c) 0 (d) None of these

6. If $U=\cot^{-1}\sqrt{\cos 2\theta}-\tan^{-1}\sqrt{\cos 2\theta}$, then $\sin U$ equals

(a) $\sin^2\theta$ (b) $\cos^2\theta$

(c) $\tan^2\theta$ (d) $\tan^2 2\theta$

7. If $\sin^{-1}(\sin 5)>x^2-4x$, then

(a) $x\in(2+\sqrt{9-2\pi},\infty)$

(b) $x\in(2-\sqrt{9-2\pi},2+\sqrt{9-2\pi})$

(c) $x\in(2-\sqrt{9-2\pi},\infty)$

(d) None of the above

8. The number of triplets satisfying $\sin^{-1}x+\cos^{-1}y+\sin^{-1}z=2\pi$, is

(a) 0 (b) 2

(c) 1 (d) infinite

9. $\sin^{-1}\{\cos(\sin^{-1}x)\}+\cos^{-1}\{\sin(\cos^{-1}x)\}$ is equal to

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{3\pi}{4}$ (d) 0

10. $\tan\left\{\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\alpha\right\}+\tan\left\{\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\alpha\right\}$, $\alpha\neq 0$, is equal to

(a) α (b) 2α

(c) $\dfrac{2}{\alpha}$ (d) None of these

11. If $(\sin^{-1}x)^2+(\cos^{-1}x)^2=\dfrac{5\pi^2}{8}$, then x is equal to

(a) ± 1 (b) $\pm\dfrac{1}{2}$

(c) $-\dfrac{1}{\sqrt{2}}$ (d) None of these

12. The solution set of the equation $\cos^{-1}x-\sin^{-1}x=\sin^{-1}(1-x)$, is

(a) $[-1,1]$ (b) $[0,1/2]$

(c) $[-1,0]$ (d) None of these

13. If $\tan^{-1}\dfrac{a}{x}+\tan^{-1}\dfrac{b}{x}+\tan^{-1}\dfrac{c}{x}+\tan^{-1}\dfrac{d}{x}=\dfrac{\pi}{2}$, then $x^4-x^2(\Sigma ab)+abcd$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

14. If $f(x)=\cos^{-1}x+\cos^{-1}\left\{\dfrac{x}{2}+\dfrac{1}{2}\sqrt{3-3x^2}\right\}$, then $f\left(\dfrac{2}{3}\right)$ equals

(a) $\dfrac{\pi}{3}$ (b) $2\cos^{-1}\dfrac{2}{3}-\dfrac{\pi}{3}$

(c) $\dfrac{2\pi}{3}$ (d) $2\cos^{-1}\dfrac{2}{3}$

15. The trigonometric equation $\sin^{-1}x=2\sin^{-1}a$ has a solution for

(a) $|a|\le\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2}<|a|<\dfrac{1}{\sqrt{2}}$

(c) all real values of a (b) $|a|<\dfrac{1}{2}$

16. If $\alpha=2\tan^{-1}(\sqrt{2}-1)$, $\beta=3\sin^{-1}\dfrac{1}{\sqrt{2}}+\sin^{-1}\left(-\dfrac{1}{2}\right)$ and $\gamma=\cos^{-1}\dfrac{1}{3}$. Then,

(a) $\alpha<\beta<\gamma$ (b) $\alpha<\gamma<\beta$ (c) $\beta<\gamma<\alpha$ (d) $\gamma<\beta<\alpha$

17. Let (x,y) be such that $\sin^{-1}(ax)+\cos^{-1}y+\cos^{-1}(bxy)=\dfrac{\pi}{2}$. If $a=1$ and $b=2$, then (x,y) lies on

(a) $x^2+y^2=1$ (b) $(x^2-1)(y^2-1)=0$

(c) $y=x$ (d) $(4x^2-1)(y^2-1)=0$

18. The value of $\cot\left[\sum_{n=1}^{23}\left\{\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right\}\right]$ is

(a) $\dfrac{23}{25}$ (b) $\dfrac{25}{23}$ (c) $\dfrac{23}{24}$ (d) $\dfrac{24}{23}$

To be Continued at Page 30

TRIGONOMETRIC EQUATION

An equation involving one or more trigonometric functions of unknown angle is called trigonometric equation.

e.g. $\sin x + \cos 2x = 1,\ \sin x = \frac{x}{10}$

Note Trigonometry identity is satisfied for every value of the unknown angle, where as trigonometric equation is satisfied for finite or infinite values of unknown angle.

Solution of Trigonometric Equation

The value of unknown angle which satisfies the given trigonometric equation, is called solution of given equation. There are three types of solutions.

Particular Solution

A specific value of unknown angle satisfying the trigonometric equation, is called its particular solution.

Principal Solution

The solutions of a trigonometric equation lie in the interval $[0, 2\pi)$ are called principal solutions.

e.g. Solution of $\sin\theta = \frac{1}{2}$ in $[0, 2\pi)$, are $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Thus, $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are the principal solutions for $\sin\theta = \frac{1}{2}$.

General Solution

Complete set of values of the unknown angle satisfying the trigonometric equation is called its general solution.

e.g. General solution of $\cos\theta = 1$ is $\theta = 2n\pi$.

General Solutions of Some Standard Trigonometric Equations

	Trigonometric equation	General solution
(i)	$\sin\theta = 0$	$\theta = n\pi,\ n \in I$
(ii)	$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2},\ n \in I$
(iii)	$\tan\theta = 0$	$\theta = n\pi,\ n \in I$
(iv)	$\sin\theta = 1$	$\theta = (4n+1)\frac{\pi}{2},\ n \in I$
(v)	$\sin\theta = -1$	$\theta = (4n-1)\frac{\pi}{2},\ n \in I$
(vi)	$\cos\theta = 1$	$\theta = 2n\pi,\ n \in I$
(vii)	$\cos\theta = -1$	$\theta = (2n+1)\pi,\ n \in I$
(viii)	$\cot\theta = 0$	$\theta = (2n+1)\frac{\pi}{2},\ n \in I$
(ix)	$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha,\ n \in I$
(x)	$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha,\ n \in I$
(xi)	$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha,\ n \in I$
(xii)	$\sin^2\theta = \sin^2\alpha$ $\cos^2\theta = \cos^2\alpha$ $\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha,\ n \in I$

REMEMBER

	Trigonometric functions	Periods
•	$\sin^a x, \cos^a x, \sec^a x, \operatorname{cosec}^a x$	$\begin{cases}\pi; \text{ if } a \text{ is even}\\ 2\pi; \text{ if } a \text{ is odd or fraction}\end{cases}$
•	$\tan^a x, \cot^a x$	π ; a is even or odd
•	$\lvert\sin x\rvert, \lvert\cos x\rvert, \lvert\tan x\rvert, \lvert\cot x\rvert, \lvert\operatorname{cosec} x\rvert, \lvert\sec x\rvert$	π
•	$\lvert\sin(ax+b)\rvert, \lvert\cos(ax+b)\rvert, \lvert\sec(ax+b)\rvert, \lvert\operatorname{cosec}(ax+b)\rvert, \lvert\tan(ax+b)\rvert, \lvert\cot(ax+b)\rvert$	$\frac{\pi}{a}$

SPEED BOOSTER

Problem solving strategy, changing a variable in trigonometric equation

- Equations of the form $P(\sin x \pm \cos x, \sin x \cos x) = 0$, where $P(x, z)$ is a polynomial, can be solved by changing $\cos x \pm \sin x = t$ $\Rightarrow 1 \pm 2\sin x \cos x = t^2$
  e.g. The equation $\sin x + \cos x = 1 + \sin x \cos x$ can be solved by substituting $\sin x + \cos x = t$.
- Equation of the form $a\sin x + b\cos x + d = 0$, where a, b and d are real numbers and $a, b \neq 0$, can be solved by changing $\sin x$ and $\cos x$ into their corresponding tangent of half the angle. e.g. The equation $3\cos x + 4\sin x = 5$ can be solved by substituting
$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \text{ and } \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$
- Many equations can be solved by introducing a new variable.
  e.g. The equation $\sin^4 2x + \cos^4 2x = \sin 2x \cos 2x$ changes to $2(y+1)\left(y - \frac{1}{2}\right) = 0$ by substituting $\sin 2x \cos 2x = y$.

IMPORTANT POINTS

Some important points to be consider while solving trigonometric equations :

- Check the validity of the given equation, e.g. $3\sin\theta - 4\cos\theta = 7$ can never be true for any θ as the value $(3\sin\theta - 4\cos\theta)$ can never exceeds $\sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$.
  So, there is no solution of this equation.
- Trigonometric equations involving $\sec\theta$ or $\tan\theta$ can never have a solution of the form $(2n+1)\pi/2$.
  Similarly, trigonometric equations involving $\operatorname{cosec}\theta$ or $\cot\theta$ can never have a solution of the form $\theta = n\pi$. The corresponding functions are undefined at these values of θ.
- Sometimes the equation has some limitations also.
  e.g. $\cot^2 x + \operatorname{cosec}^2 x = 1$ can be true only if $\cot^2 x = 0$ and $\operatorname{cosec}^2 x = 1$, simultaneously as $\operatorname{cosec}^2 x \geq 1$.
  Hence, the solution is $\theta = (2n+1)\pi/2$.

Trigonometric Inequalities

For solving trigonometric inequation $f(x) \le a$ or $f(x) \ge a$, where $f(x)$ is some trigonometric function, the following steps should be taken

(i) Draw the graph of $f(x)$ in an interval length equal to the fundamental period of $f(x)$.

(ii) Draw the line $y = a$.

(iii) Take the portion of the graph for which the inequation is satisfied.

(iv) To generalise, add nP $(n \in I)$ and take union over the set of integers; where P is fundamental period of $f(x)$.

MASTER STROKES

1. The set of values of a for which the equation $\sin x(\sin x + \cos x) = 0$ has real solutions, is

(a) $[1 - \sqrt{2}, 1 + \sqrt{2}]$ *(b)* $[2 - \sqrt{3}, 2 + \sqrt{3}]$

(c) $[0, 2 + \sqrt{3}]$ *(d)* $\left[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]$

2. The solution set of $\sin\left(x - \frac{\pi}{4}\right) - \cos\left(x + \frac{3\pi}{4}\right) = 1$ and $\frac{2\cos 7x}{\cos 3 + \sin 3} = 2^{\cos 2x}$ is

(a) $k\pi + (-1)^4 \frac{\pi}{4}, k \in Z$ *(b)* $(8k + 3)\frac{\pi}{4}, k \in Z$

(c) $(8k + 1)\frac{\pi}{4}, k \in Z$ *(d)* None of these

3. The solution set of the equation $(\sin 13^\circ)^{\cot 3x + \cot x} = \sin^2(4\pi - x) - \cos(3\pi - x)\cos(2\pi + x)$ is given by

(a) $x = \frac{n\pi}{4}, n \ne 4\lambda, \lambda \in Z$ *(b)* $x = \frac{n\pi}{2}, n \ne 2\lambda, \lambda \in Z$

(c) $x = \frac{n\pi}{3}, n \ne 3\lambda$ *(d)* None of these

4. The number of real solutions of the equation $\sin e^x \cos e^x = 2^{x-2} + 2^{-x-2}$, is

(a) 0 *(b)* 1 *(c)* 2 *(d)* infinite

5. If $\sin x = \lambda$ has exactly one solution in $\left[0, \frac{9\pi}{4}\right]$, then the number of values of λ, is

(a) 0 *(b)* 1 *(c)* 2 *(d)* 3

6. Number of solution(s) of the equation $\left(\frac{2 - \cos^2 x}{\sin x}\right)^3 + \left(\frac{3 - \cos 2x}{\sin x}\right) = 0$, is

(a) 0 *(b)* 1 *(c)* 2 *(d)* infinite

7. The set of values of x satisfying $[\sin(\cos x)] = -1$, is ($[\cdot]$ denotes the greatest integer function)

(a) $[(4n + 1)\pi, (4n + 3)\pi], n \in Z$

(b) $\left[(n + 1)\frac{\pi}{2}, (4n + 3)\frac{\pi}{2}\right], n \in Z$

(c) $\left((4n + 1)\frac{\pi}{2}, (4n + 3)\frac{\pi}{2}\right), n \in Z$

(d) None of the above

8. The solution set of the inequation $\cos x - \sin x \ge 1$ in $[0, 2\pi]$ is

(a) $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right]$ *(b)* $\left[\frac{3\pi}{2}, \frac{7\pi}{4}\right] \cup \{0\}$

(c) $\left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$ *(d)* None of these

9. If $0 \le x, y \le 2\pi$ and $\sin x + \sin y = 2$, then $x + y$ is equals to

(a) π *(b)* $\frac{\pi}{2}$

(c) 3π *(d)* None of these

10. The most general values of θ satisfying $2\cos\theta + \sqrt{3} = 0$ and $\sqrt{3}\tan\theta - 1 = 0$ are given by

(a) $n\pi \pm \frac{\pi}{6}, n \in Z$ *(b)* $2n\pi \pm \frac{\pi}{6}, n \in Z$

(c) $2n\pi + \frac{7\pi}{6}, n \in Z$ *(d)* None of these

11. The number of values of x in $[0, 4\pi]$ satisfying the inequation $|\sqrt{3}\cos x - \sin x| \ge 2$, is

(a) 0 *(b)* 2 *(c)* 4 *(d)* 8

12. The number of solutions of the equation $\sin x = |\cos 3x|$ in $[0, \pi]$ is

(a) 3 *(b)* 4 *(c)* 5 *(d)* 6

13. The number of solutions of the equation $\sin x = [1 + \sin x] + [1 - \cos x]$ in $[0, 2\pi]$, is

(a) 0 *(b)* 1 *(c)* 2 *(d)* 6

14. If the values of x between 0 and 2π which satisfy the equation $\sin x|\cos x| = \frac{1}{2\sqrt{2}}$ are in AP, then the common difference of the AP, is

(a) $\frac{\pi}{8}$ *(b)* $\frac{\pi}{4}$ *(c)* $\frac{3\pi}{8}$ *(d)* $\frac{5\pi}{8}$

15. The number of solutions of the equation $2\tan x + x = \frac{12\pi}{5}$ in $[0, 2\pi]$, is

(a) 1 *(b)* 2 *(c)* 3 *(d)* infinite

16. The roots of the equation $2\sin^2\theta + \sin^2 2\theta = 2$ are

(a) $\frac{\pi}{6}$ *(b)* $\frac{\pi}{4}$ *(c)* $\frac{\pi}{3}$ *(d)* $\frac{\pi}{2}$

17. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of $\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + (m - 1)\frac{\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is (are)

(a) $\frac{\pi}{4}$ *(b)* $\frac{\pi}{6}$ *(c)* $\frac{\pi}{12}$ *(d)* $\frac{5\pi}{12}$

18. The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is

(a) 0 *(b)* 1

(c) 2 *(d)* 4

19. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$, is
(a) 2 *(b)* 5 *(c)* 7 *(d)* 3

20. The set of values of θ satisfying the inequation $2\sin^2\theta - 5\sin\theta + 2 > 0$, where $0 < \theta < 2\pi$, is
(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ *(b)* $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$
(c) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, 2\pi\right]$ *(d)* None of these

21. The number of distinct solutions of the equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$, is
(a) 4 *(b)* 6 *(c)* 8 *(d)* 5

22. The possible values of $\theta \in (0, \pi)$ such that $\sin\theta + \sin 4\theta + \sin 7\theta = 0$, are
(a) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{6}$ *(b)* $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$
(c) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$ *(d)* $\frac{\pi}{4}, \frac{5\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

23. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has
(a) infinitely many solutions *(b)* three solutions
(c) one solution *(d)* no solution

24. The values of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\frac{3}{5}\right)$ are
(a) ± 4 *(b)* 0, 4 *(c)* $-4, 0$ *(d)* 4, 5

25. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations
$(y+z)\cos 3\theta = xyz \sin 3\theta$
$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$
$xyz \sin 3\theta = (y + 2z)\cos 3\theta + y \sin 3\theta$
have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is
(a) 0 *(b)* 2 *(c)* 3 *(d)* 4

Answers

1. (d)	2. (b)	3. (a)	4. (a)	5. (c)
6. (a)	7. (c)	8. (c)	9. (a)	10. (c)
11. (c)	12. (d)	13. (a)	14. (b)	15. (b)
16. (b,d)	17. (c,d)	18. (c)	19. (d)	20. (b)
21. (c)	22. (c)	23. (d)	24. (a)	25. (c)

Continued from Page 27

19. If $(\sin^{-1}x + \sin^{-1}y)(\sin^{-1}z + \sin^{-1}w) = \pi^2$ and $n_1, n_2, n_3, n_4 \in N$, then the value of $\begin{vmatrix} x^{n_1} & y^{n_2} \\ z^{n_3} & w^{n_4} \end{vmatrix}$ cannot be equal to
(a) -2 *(b)* 0 *(c)* 1 *(d)* 2

20. If $0 < x < 1$, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2}$ equals
(a) $\frac{x}{\sqrt{1+x^2}}$ *(b)* x
(c) $x\sqrt{1+x^2}$ *(d)* $\sqrt{1+x^2}$

21. If $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{2r^2}\right) = t$, then $\tan t$ is equal to
(a) $\frac{2}{3}$ *(b)* 1
(c) $\frac{\sqrt{5}}{3}$ *(d)* None of these

22. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then, the value of y is
(a) $\frac{3x+x^2}{1-3x^2}$ *(b)* $\frac{3x+x^2}{1+3x^2}$ *(c)* $\frac{3x-x^3}{1-3x^2}$ *(d)* $\frac{3x+x^3}{1-3x^2}$

23. The number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is
(a) 1 *(b)* 2
(c) 8 *(d)* 9

24. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, then $x^4 + y^4 + z^4 + 4x^2y^2z^2 = k(x^2y^2 + y^2z^2 + z^2x^2)$, where k equals
(a) 1 *(b)* 2
(c) 4 *(d)* None of these

25. Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is
(a) 2 *(b)* 3
(c) 4 *(d)* 5

Answers

1. (a)	2. (b)	3. (b)	4. (c)	5. (b)
6. (c)	7. (b)	8. (b)	9. (b)	10. (c)
11. (c)	12. (d)	13. (b)	14. (a)	15. (a)
16. (b)	17. (a)	18. (b)	19. (c)	20. (c)
21. (b)	22. (c)	23. (a)	24. (b)	25. (b)

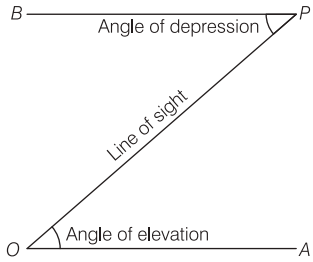
HEIGHT AND DISTANCE

Sometimes we are required to find the height of a tower, tree, building and distance of a ship from light house, width of a river, etc. We cannot measure them accurately, though we can find them using the knowledge of trigonometric ratio.

Angles of Elevation and Depression

Let O and P be two points such that the point P is at higher level. Let OA and PB be horizontal lines through O and P, respectively.

If an observer is at O and the point P is the object under consideration, then the line OP is called the line of sight of the point P and the $\angle AOP$, between the line of sight and the horizontal line OA, is known as the angle of elevation of point P as seen from O.



If an observer is at P and the object under consideration is at O, then $\angle BPO$ is known as the angle of depression of O as seen from P.

Obviously, the angle of elevation of a point P as seen from a point O is equal to the angle of depression of O as seen from P.

Some Useful Results

(i) Any line perpendicular to a plane is perpendicular to every line lying in the plane.

(ii) In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle.

(iii) In an isosceles triangle the median is perpendicular to the base.

(iv) Angles in the same segment of a circle are equal.

MASTER STROKES

1. A man from the top of a 100 m high tower sees a car moving towards the tower at an angle of depression of 30°. After sometime, the angle of depression becomes 60°. The distance (in metres) travelled by the car during this time is

(a) $100\sqrt{3}$ *(b)* $\frac{200\sqrt{3}}{3}$ *(c)* $\frac{100\sqrt{3}}{3}$ *(d)* $200\sqrt{3}$

2. A pole stands vertically, inside a triangular park ΔABC. If the angle of elevation of the top of the pole from each corner of the park is same, then in ΔABC the foot of the pole is at the

(a) centroid *(b)* circumcentre

(c) incentre *(d)* orthocentre

3. At a point 15 m away from the base of a 15 m high house, the angle of elevation of the top is

(a) 90° *(b)* 60° *(c)* 30° *(d)* 45°

4. From the bottom of a pole of height h, the angle of elevation of the top of a tower is α. The pole subtends an angle β at the top of the tower. The height of the tower is

(a) $\frac{h\sin\alpha\sin(\alpha-\beta)}{\sin\beta}$ *(b)* $\frac{h\sin\alpha\cos(\alpha+\beta)}{\cos\beta}$

(c) $\frac{h\sin\alpha\cos(\alpha-\beta)}{\sin\beta}$ *(d)* $\frac{h\sin\alpha\sin(\alpha+\beta)}{\cos\beta}$

5. The lengths of the shadows of a vertical pole of height h, thrown by the Sun's rays at three different moments are h, $2h$ and $3h$. The sum of the angles of elevation of the rays at these three moments is equal to

(a) $\frac{\pi}{2}$ *(b)* $\frac{\pi}{3}$ *(c)* $\frac{\pi}{4}$ *(d)* $\frac{\pi}{6}$

6. Three vertical poles of heights h_1, h_2 and h_3 at the vertices A, B and C of a ΔABC subtend angles α, β and γ respectively at the circumcentre of the triangle. If $\cot\alpha$, $\cot\beta$ and $\cot\gamma$ are in AP, then h_1, h_2, h_3 are in

(a) AP *(b)* GP

(c) HP *(d)* None of these

7. The angles of elevation of the top of a tower from the top and bottom of a building of height 'a' are 30° and 45°, respectively. If the tower and the building stand at the same level, then the height of the tower is

(a) $\frac{a(3+\sqrt{3})}{2}$ *(b)* $a(\sqrt{3}+1)$ *(c)* $\sqrt{3}a$ *(d)* $a(\sqrt{3}-1)$

8. The angle of elevation of the top of a hill from each of the vertices A, B and C of a horizontal triangle is α. The height of the hill is

(a) $b\tan\alpha\,\mathrm{cosec}\,B$ *(b)* $\frac{1}{2}a\tan\alpha\,\mathrm{cosec}\,A$

(c) $\frac{1}{2}c\tan\alpha\,\mathrm{cosec}\,C$ *(d)* None of these

9. The angular elevation of a tower OP at a point A due South of it is 60° and at a point B due West of A, the elevation is 30°. If $AB = 3$ m, then the height of the tower is

(a) $2\sqrt{3}$ m *(b)* $2\sqrt{6}$ m *(c)* $\frac{3\sqrt{3}}{2}$ m *(d)* $\frac{3\sqrt{6}}{4}$ m

10. A vertical pole PO is standing at the centre O of a square $ABCD$. If AC subtends an angle of 90° at the top, P of the pole, then the angle subtended by a side of the square at P is

(a) 45° *(b)* 30°

(c) 60° *(d)* None of these

11. Let α be the solution of $16^{\sin^2\theta} + 16^{\cos^2\theta} = 10$ in $(0, \pi/4)$. If the shadow of a vertical pole is $\frac{1}{\sqrt{3}}$ of its height, then the altitude of the sun is

(a) α *(b)* $\frac{\alpha}{2}$ *(c)* 2α *(d)* $\frac{\alpha}{3}$

12. A circular ring of radius 3 cm is suspended horizontally from a point 4 cm vertically above the centre by 4 strings attached at equal intervals to its circumference. If the angle between two consecutive strings be θ, then $\cos\theta$ equals

(a) $\frac{4}{5}$ *(b)* $\frac{4}{25}$ *(c)* $\frac{16}{25}$ *(d)* None of these

13. The shadow of a pole of height $(\sqrt{3}+1)$ m standing on the ground is found to be 2 m longer when the elevation is 30° than when elevation was α. Then, α is equal to

(a) 75° *(b)* 60° *(c)* 45° *(d)* 30°

14. A vertical lamp-post of height 9 m stands at the corner of a rectangular field. The angle of elevation of its top from the farthest corner is 30°, while from another corner it is 45°. The area of the field is

(a) $81\sqrt{2}$ m^2 *(b)* $9\sqrt{2}$ m^2 *(c)* $81\sqrt{3}$ m^2 *(d)* $9\sqrt{3}$ m^2

15. An isosceles triangle of wood of base $2a$ and height h is placed with its base on the ground and vertex directly above. The triangle faces the sun whose altitude is 30°. Then, the tangent of the angle at the apex of the shadow is

(a) $\frac{2ah}{\sqrt{3}}$ *(b)* $\frac{2\sqrt{3}\,ah}{3h^2 - a^2}$ *(c)* $\frac{a^2 + h^2}{2\sqrt{3}}$ *(d)* $\frac{2\sqrt{3}\,ah}{3h^2 + a^2}$

16. A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that $AB(= a)$ subtends an angle of 60° at the foot of the tower, and the angle of elevation of the top of the tower from A or B is 30°. The height of the tower is

(a) $\frac{a}{\sqrt{3}}$ *(b)* $\sqrt{3}\,a$ *(c)* $\frac{2a}{\sqrt{3}}$ *(d)* $2\sqrt{3}\,a$

17. The angle of elevation of the top of a hill from a point is α. After walking b m towards the top up of a slope inclined at an angle β to the horizon, the angle of elevation of the top becomes γ. Then, the height of the hill is

(a) $\frac{b\sin\alpha\sin(\gamma-\beta)}{\sin(\gamma-\alpha)}$ *(b)* $\frac{b\sin\alpha\sin(\gamma-\alpha)}{\sin(\gamma-\beta)}$

(c) $\frac{b\sin(\gamma-\beta)}{\sin(\gamma-\alpha)}$ *(d)* $\frac{\sin(\gamma-\beta)}{b\sin\alpha\sin(\gamma-\alpha)}$

18. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60°. He moves away from the pole along the line BC to a point D such that $CD = 7$ m. From D the angle of elevation of the point A is 45°. Then, the height of the pole is

(a) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)$ m *(b)* $\frac{7\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}+1}\right)$m

(c) $\frac{7\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}-1}\right)$m *(d)* $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$ m

19. A house subtends a right angle at the window of an opposite house and the angle of elevation of the window from the bottom of the first house is 60°. If the distance between two houses is 6 m, then the height of the first house is

(a) $8\sqrt{3}$ m *(b)* $6\sqrt{3}$ m

(c) $4\sqrt{3}$ m *(d)* None of these

20. The angle of elevation of top of a tower from a point on the ground is 30° and it is 60° when it is viewed from a point located 40 m away from the initial point towards the tower, then the height of the tower is

(a) $20\sqrt{3}$ m *(b)* $\frac{\sqrt{3}}{20}$ m *(c)* $10\sqrt{3}$ m *(d)* $\frac{\sqrt{3}}{10}$ m

21. From the top of a hill h m high, the angles of depression of the top and the bottom of a pillar are α and β, respectively. The height (in metre) of the pillar is

(a) $\frac{h(\tan\beta - \tan\alpha)}{\tan\beta}$ *(b)* $\frac{h(\tan\alpha - \tan\beta)}{\tan\alpha}$

(c) $\frac{h(\tan\beta + \tan\alpha)}{\tan\beta}$ *(d)* $\frac{h(\tan\beta + \tan\alpha)}{\tan\alpha}$

22. A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then, the speed (in m/s) of the bird is

(a) $20\sqrt{2}$ *(b)* $20(\sqrt{3}-1)$

(c) $40(\sqrt{2}-1)$ *(d)* $40(\sqrt{3}-\sqrt{2})$

23. If the angles of elevation of the top of a tower from three collinear points A, B and C, on a line leading to the foot of the tower are 30°, 45° and 60° respectively, then the ratio $AB : BC$, is

(a) $1 : \sqrt{3}$ *(b)* $2 : 3$ *(c)* $\sqrt{3} : 1$ *(d)* $\sqrt{3} : \sqrt{2}$

24. The angle of elevation of the top of a vertical pole when observed from each vertex of a regular hexagon is $\frac{\pi}{3}$. If the area of the circle circumscribing the hexagon is A m^2, then the area of the hexagon is

(a) $\frac{3\sqrt{3}\,A}{8}$ m^2 *(b)* $\frac{\sqrt{3}\,A}{\pi}$ m^2

(c) $\frac{3\sqrt{3}\,A}{4\pi}$ m^2 *(d)* $\frac{3\sqrt{3}\,A}{2\pi}$ m^2

25. The upper $\left(\frac{3}{4}\right)$th portion of a vertical pole subtends an angle $\tan^{-1}\frac{3}{5}$ at a point in the horizontal plane through its foot at a distance 40 m from the foot. A possible height of the vertical pole is

(a) 20 *(b)* 40 *(c)* 60 *(d)* 80

Answers

1. (b)	**2.** (b)	**3.** (d)	**4.** (c)	**5.** (a)
6. (a)	**7.** (a)	**8.** (b)	**9.** (d)	**10.** (c)
11. (c)	**12.** (c)	**13.** (c)	**14.** (a)	**15.** (b)
16. (a)	**17.** (a)	**18.** (d)	**19.** (a)	**20.** (a)
21. (a)	**22.** (b)	**23.** (c)	**24.** (d)	**25.** (b)

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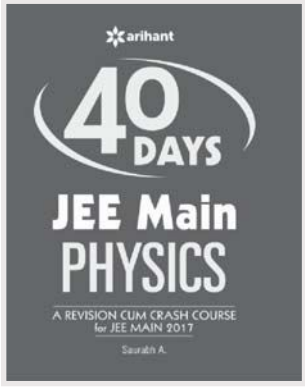

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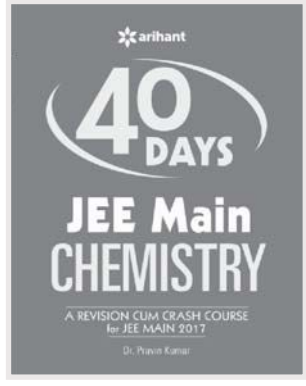

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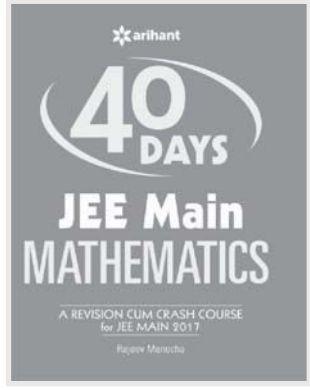

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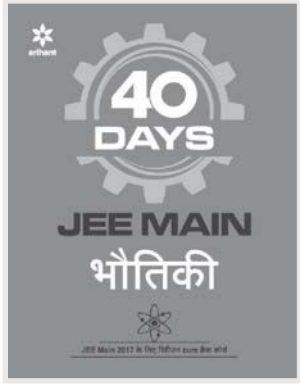

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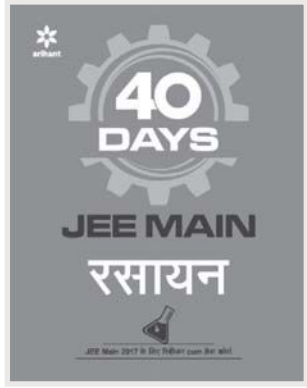

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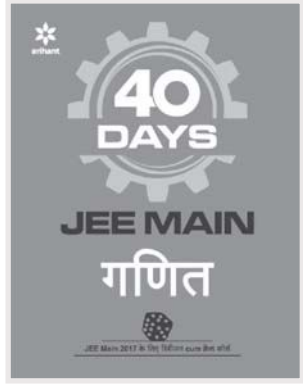

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Mock Questions from Complete Syllabus with Complete Solutions

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. The value of the determinant $\begin{vmatrix} 8! & 9! & 10! \\ 9! & 10! & 11! \\ 10! & 11! & 12! \end{vmatrix}$ is

(a) (8!) (9!) (10!) *(b)* (8) (9) (10)
(c) 2(8!) (9!) (10!) *(d)* 4(8!) (9!) (10!)

2. The sum of the series $1 + \frac{5}{6} + \frac{15}{6^2} + \frac{25}{6^3} + \frac{35}{6^4} + \ldots$ upto infinity is

(a) $\frac{5}{12}$ *(b)* $\frac{7}{12}$ *(c)* $\frac{12}{7}$ *(d)* $\frac{12}{5}$

3. $\lim\limits_{\theta \to 0} \frac{5^{\tan\theta} - 5^{\sin\theta}}{\tan\theta - \sin\theta}$ is equal to

(a) log 4 *(b)* 1 *(c)* log 5 *(d)* 5

4. Let $f(x) = ||x| - 3|$, then points, where $f(x)$ is not differentiable, is (are)

(a) $\pm$ 3 *(b)* 0, $\pm$ 3 *(c)* 3 *(d)* 0

5. If Q is the reflection of $P(3, 6)$ in Y-axis and R is the reflection of Q in X-axis, then the distance between P and R is

(a) $4\sqrt{5}$ *(b)* $\sqrt{5}$
(c) $6\sqrt{5}$ *(d)* $3\sqrt{5}$

6. If O is the origin and P is a point on the locus $y^2 = 16x$. Then, the locus of the middle point of OP is

(a) $y^2 = 8x$ *(b)* $y^2 = 4x$
(c) $y^2 = 2x$ *(d)* $y^2 = 16x$

7. The equation of the tangent to the circle $x^2 + y^2 + 8x - 8y + 16 = 0$, which makes equal intercepts on the positive coordinate axes, is

(a) $x + y = 2\sqrt{2}$ *(b)* $x + y = 5\sqrt{2}$
(c) $x + y = 4\sqrt{2}$ *(d)* $x + y = 1$

8. The minimum area of triangle formed by the tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and coordinate axes, is

(a) 20 sq units *(b)* 15 sq units
(c) 25 sq units *(d)* None of these

9. If a parabola has the origin as its focus and the line $x = 4$ as the directrix. Then, the vertex of the parabola is at

(a) (0, 2) *(b)* (2, 0) *(c)* (0, 4) *(d)* (2, 2)

10. If the line passes through the points $(5, 1, p)$ and $(3, q, 1)$ crosses the YZ-plane at the point $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$. Then,

(a) $p = 4, q = 6$ *(b)* $p = 6, q = 4$
(c) $p = 8, q = 2$ *(d)* $p = 2, q = 8$

11. If $\left|z - \frac{2}{z}\right| = 2$, then the maximum value of $|z|$ is equal to

(a) $\sqrt{3} + 1$ *(b)* $\sqrt{3} - 1$
(c) $\sqrt{5} + 1$ *(d)* 4

12. Angle between the tangent to the curve $y = x^2 - 7x + 9$ at the point (3, 0) and (4, 0) is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

13. The domain of definition of the function $y(x)$ given by the equation $5^x + 5^y = 5$, is

(a) $-5 < x < 5$ (b) $0 < x < 1$
(c) $-\infty < x < 0$ (d) $-\infty < x < 1$

14. $\int (x^6 + 1)^{-1} e^{5 \log x} dx$ is equal to

(a) $\frac{1}{6}\log(1 + x^6) + C$ (b) $\frac{1}{5}\log(1 + x^5) + C$
(c) $\log(1 + x^6) + C$ (d) $\log(1 + x^5) + C$

15. If $x^2 + 2ax - 4a + 21 > 0, \forall\ x \in R$, then

(a) $a < -7$ (b) $a > 7$
(c) $-7 < a < 3$ (d) $3 < a < 7$

16. If $P = \begin{bmatrix} 1 & x \\ x^2 & 5y \end{bmatrix}$, $Q = \begin{bmatrix} -7 & 3 \\ 1 & 0 \end{bmatrix}$ and $\text{adj}(P) + Q = \begin{bmatrix} 8 & 0 \\ -8 & 1 \end{bmatrix}$, then the value of x and y are

(a) 1, 1 (b) 3, 4 (c) 3, 3 (d) 3, – 3

17. The average marks of boys in a class is 62 and that of girls is 54. The average marks of boys and girls combined is 60. The percentage of boys in the class is

(a) 40% (b) 50% (c) 80% (d) 75%

18. The horizontal distance between two towers is 90 m and the angle of depression of the top of the first tower as seen from the top of the second is 30°. If the height of the second tower is 180 m, then the height of the first tower is

(a) $(180 + 30\sqrt{3})$ m (b) $(90 + \sqrt{3})$ m
(c) $(30 + 30\sqrt{3})$ m (d) None of these

19. Which of the following is the contrapositive of "If two triangles are identical, then these are similar"?

(a) If two triangles are not similar, then these are identical
(b) If two triangles are not identical, then these are similar
(c) If two triangles are not similar, then these are not identical
(d) If two triangles are not identical, then these are not similar

20. If $\log_{10} 2$, $\log_{10}(5^x - 1)$ and $\log_{10}(5^x + 3)$ are three consecutive terms of an AP, then

(a) $x = 0$ (b) $x = 1$
(c) $x = \log_{10} 5$ (d) None of these

21. The sum of the coefficient in the expansion of $(1 + 5x - 7x^2)^{3546}$ is

(a) 275 (b) 1 (c) 3546 (d) 575

22. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$, is

(a) $3x + 2y - 2z = 0$ (b) $x + 2y - 2z = 0$
(c) $5x + 2y - 4z = 0$ (d) $x - 2y + z = 0$

23. $|\hat{\mathbf{i}} + \mathbf{p}|^2 + |\hat{\mathbf{j}} \times \mathbf{p}|^2 + |\hat{\mathbf{k}} \times \mathbf{p}|^2$ is equal to

(a) $4p^2$ (b) $2p^2$ (c) p^2 (d) $\frac{p^2}{2}$

24. There are eight machines and it is known that exactly three of them are faulty. They are tested, one-by-one, in a random order till all the faulty machines are identified. Then, the probability that only three tests are needed, is

(a) $\frac{1}{56}$ (b) $\frac{1}{6}$
(c) $\frac{1}{36}$ (d) None of these

25. How many signals that can be sent by using 7 flags of different colours, taking one or more at a time?

(a) 12655 (b) 13699 (c) 17699 (d) 13299

26. If $\int_{-4}^{6} f(x)\,dx = 6$ and $\int_{3}^{6}\{4 - f(x)\}\,dx = 15$, then the value of $\int_{-4}^{3} f(x)\,dx$ is

(a) 1 (b) 8 (c) 9 (d) 6

27. If the area between the curves $x = py^2$ and $y = px^2$ $(p > 0)$ is 3 sq units. Then, the value of p is

(a) $\frac{1}{\sqrt{3}}$ (b) $\sqrt{3}$ (c) 1 (d) $\frac{1}{3}$

28. If $\sin^2 x$ is an integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$, then P can be

(a) $2\sin x$ (b) $2\cot x$ (c) $\cos x$ (d) $\cot x$

29. If $f(x) = [\tan^2 x]$, where $[\cdot]$ denotes the greatest integer function, then

(a) $f(x)$ is differentiable at $x = 0$
(b) $f(x)$ is continuous at $x = 0$
(c) $\lim_{x \to 0} f(x)$ does not exist
(d) $f(x)$ is not differentiable at $x = 0$

30. $\tan^{-1}\frac{3\sin 2\alpha}{5 + 3\cos 2\alpha} + \tan^{-1}\frac{\tan\alpha}{4}$ is equal to $\left[\text{where, } -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right]$

(a) α (b) 2α
(c) 3α (d) 4α

DO YOU Know?

In a group of 23 people, atleast two have the same birthday with the probability greater than 1/2.

Answers with Explanation

1. *(c)* Let $\Delta = \begin{vmatrix} 8! & 9! & 10! \\ 9! & 10! & 11! \\ 10! & 11! & 12! \end{vmatrix}$

$$= (8!)\,(9!)\,(10!)\begin{vmatrix} 1 & 9 & 9\times 10 \\ 1 & 10 & 10\times 11 \\ 1 & 11 & 11\times 12 \end{vmatrix}$$

On applying operations

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = (8!)\,(9!)\,(10!)\begin{vmatrix} 1 & 9 & 9\times 10 \\ 0 & 1 & 20 \\ 0 & 2 & 42 \end{vmatrix}$$

$$= (8!)\,(9!)\,(10!)\,[1(42-40)] = 2\,(8!)\,(9!)\,(10!)$$

2. *(d)* Let $S = 1 + \frac{5}{6} + \frac{15}{6^2} + \frac{25}{6^3} + \frac{35}{6^4} + \ldots$

$$\Rightarrow \quad S - 1 = \frac{5}{6} + \frac{15}{6^2} + \frac{25}{6^3} + \ldots \qquad \ldots(i)$$

$$\Rightarrow \quad \frac{S-1}{6} = \frac{5}{6^2} + \frac{15}{6^3} + \frac{25}{6^4} + \ldots \qquad \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{5}{6}(S-1) = \frac{5}{6} + \frac{10}{6^2} + \frac{10}{6^3} + \frac{10}{6^4} + \ldots$$

$$\Rightarrow \quad S - 1 = 1 + \frac{2}{6} + \frac{2}{6^2} + \frac{2}{6^3} + \ldots$$

$$\therefore \quad S = 2 + \frac{2}{6}\left[1 + \frac{1}{6} + \frac{1}{6^2} + \ldots\right]$$

$$= 2 + \frac{\frac{2}{6}}{1 - \frac{1}{6}} = 2 + \frac{2}{6}\times\frac{6}{5} = \frac{12}{5}$$

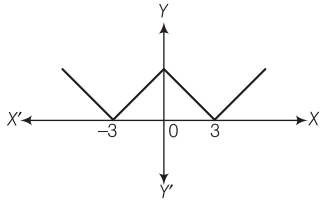
3. *(c)* Let $L = \lim\limits_{\theta\to 0}\dfrac{5^{\tan\theta} - 5^{\sin\theta}}{\tan\theta - \sin\theta}$

$$= \lim_{\theta\to 0}\frac{(5^{\tan\theta} - 1) - (5^{\sin\theta} - 1)}{\tan\theta - \sin\theta}$$

$$= \lim_{\theta\to 0}\frac{\frac{5^{\tan\theta}-1}{\tan\theta}\cdot\tan\theta - \frac{5^{\sin\theta}-1}{\sin\theta}\cdot\sin\theta}{\tan\theta - \sin\theta}$$

$$= \lim_{\theta\to 0}\log 5\cdot\frac{\tan\theta - \sin\theta}{\tan\theta - \sin\theta} = \log 5$$

4. *(b)* Graph of $f(x) = ||x| - 3|$

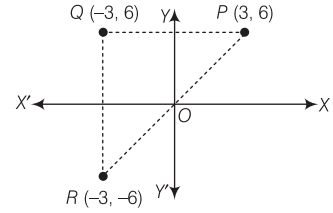


In the above graph, there are three sharp edges at

$$x = -3, 0, 3$$

Hence, $f(x)$ is not differentiable at $\{-3, 0, 3\}$.

5. *(c)* We can depict given information in question by following figure



$\therefore$ Distance between P and R is

$$|PR| = \sqrt{(-3-3)^2 + (-6-6)^2} = \sqrt{36+144} = \sqrt{180} = 6\sqrt{5}$$

6. *(a)* Let coordinates of $P \equiv (x_1, y_1)$ and $R(\alpha, \beta)$ be the moving point.

According to the question, $P(x_1, y_1)$ lies on the locus $y^2 = 16x$

$$\therefore \quad y_1^2 = 16x_1 \qquad \ldots(i)$$

Also, $R(\alpha, \beta)$ is the middle point of OP.

$$\therefore \quad \alpha = \frac{0 + x_1}{2} \text{ and } \beta = \frac{0 + y_1}{2}$$

or $\quad x_1 = 2\alpha$ and $y_1 = 2\beta$

On putting the values of x_1 and y_1 in Eq. (i), we get

$$(2\beta)^2 = 16(2\alpha) \Rightarrow 4\beta^2 = 32\alpha \Rightarrow \beta^2 = 8\alpha$$

$\therefore$ Locus of point $R(\alpha, \beta)$ is $y^2 = 8x$

7. *(c)* Given circle is $x^2 + y^2 + 8x - 8y + 16 = 0$.

Centre $\equiv (-4, 4)$ and radius $= \sqrt{(-4)^2 + (4)^2 - 16} = 4$

Let the equation of tangent, which makes equal intercepts on the positive coordinate axes be

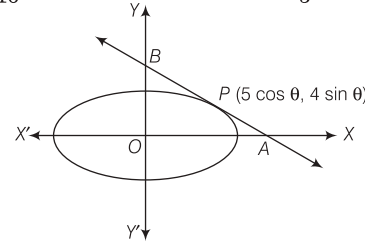
$$\frac{x}{a} + \frac{y}{a} = 1 \text{ i.e. } x + y = a$$

$\because$ Length of perpendicular from the centre $(-4, 4)$ to tangent $x + y = a$ is equal to radius of the circle.

$$\therefore \quad \frac{|-4+4-a|}{\sqrt{1+1}} = 4 \Rightarrow a = 4\sqrt{2}$$

Hence, the equation of the tangent is $x + y = 4\sqrt{2}$.

8. *(a)* We know that, equation of tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at $P(5\cos\theta, 4\sin\theta)$ is $\frac{x\cos\theta}{5} + \frac{y\sin\theta}{4} = 1$



Point of intersection of tangent to coordinate axes are $A(5\sec\theta, 0)$ and $B(0, 4\operatorname{cosec}\theta)$.

$$\therefore \text{ Area of } \Delta OAB = \frac{1}{2}\times OA\times OB = \frac{1}{2}\times 5\sec\theta\times 4\operatorname{cosec}\theta$$

$$= \frac{20}{2\sin\theta\cdot\cos\theta} = \frac{20}{\sin 2\theta}$$

Value of $\frac{20}{\sin 2\theta}$ will be minimum, when value of $\sin 2\theta$ is maximum, i.e. $\sin 2\theta = 1$

$\therefore$ Minimum value of area of $\Delta OAB = 20$ sq units

9. ***(b)*** Given, $x = 4$ is the directrix of the parabola and origin as focus of parabola.
We know that, the mid-point of focus and directrix is the vertex of the parabola.
$\therefore$ Vertex $= \left(\frac{0+4}{2}, 0\right) = (2, 0)$

10. ***(b)*** Equation of the line passing through $(5, 1, p)$ and $(3, q, 1)$ is

$$\frac{x-3}{5-3} = \frac{y-q}{1-q} = \frac{z-1}{p-1} \quad \ldots(i)$$

Also, point $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$ lies on Eq. (i), we get

$$\frac{-3}{2} = \frac{\frac{17}{2} - q}{1-q} = \frac{\frac{-13}{2} - 1}{p-1}$$

From Ist and IIIrd terms,

$$p - 1 = \frac{\left(\frac{-15}{2}\right)}{\left(\frac{-3}{2}\right)} \Rightarrow p = 6$$

From Ist and IInd terms,

$$-3(1-q) = 2\left(\frac{17}{2} - q\right) \Rightarrow q = 4$$

11. ***(a)*** Given, $\left|z - \frac{2}{z}\right| = 2$

$$\Rightarrow \quad |z| = \left|\left(z - \frac{2}{z}\right) + \frac{2}{z}\right|$$

$$\Rightarrow \quad |z| \le \left|z - \frac{2}{z}\right| + \frac{2}{|z|}$$

$$\Rightarrow \quad |z| \le 2 + \frac{2}{|z|} \qquad \left[\because \left|z - \frac{2}{z}\right| = 2\right]$$

$$\Rightarrow \quad |z|^2 - 2|z| - 2 \le 0$$

$$\Rightarrow \quad [|z| - (\sqrt{3}+1)][|z| - (1-\sqrt{3})] \le 0$$

$$\therefore \quad 1 - \sqrt{3} \le |z| \le \sqrt{3} + 1$$

12. ***(b)*** Given, $y = x^2 - 7x + 9$

$$\therefore \quad \frac{dy}{dx} = 2x - 7$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{(3,0)} = 2(3) - 7 = -1$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{(4,0)} = 2(4) - 7 = 1$$

Here, $m_1 = -1$ and $m_2 = 1$
and $m_1 \cdot m_2 = -1 \cdot 1 = -1$
Hence, the tangents are perpendicular to each other.

13. ***(d)*** $\because 5^x + 5^y = 5 \Rightarrow 5^y = 5 - 5^x$

$$\because \quad 5^y > 0$$

$$\therefore \quad 5 - 5^x > 0$$

$$\therefore \quad 5^x < 5^1 \Rightarrow x < 1$$

$$\Rightarrow \quad x \in (-\infty, 1) \quad \text{or} \quad -\infty < x < 1$$

14. ***(a)*** Let $I = \int (x^6 + 1)^{-1} e^{5\log x}\, dx$

$$= \int \frac{e^{5\log x}}{x^6 + 1}\, dx = \int \frac{x^5}{1 + x^6}\, dx \quad [\because e^{a\log x} = x^a]$$

$$= \frac{1}{6}\log(1 + x^6) + C$$

15. ***(c)*** We know, $x^2 + 2ax - 4a + 21 > 0, \forall\ x \in R$, when discriminant of the given quadratic expression is less than zero and coefficient of x^2 is greater than 0.
Here, sign of coefficient of x^2 is positive.

$$\therefore \quad (2a)^2 - 4(1)(21 - 4a) < 0$$

$$\Rightarrow \quad a^2 + 4a - 21 < 0$$

$$\Rightarrow \quad (a+7)(a-3) < 0$$

$$\therefore \quad -7 < a < 3$$

16. ***(c)*** $\text{adj}(P) = \begin{bmatrix} 5y & -x \\ -x^2 & 1 \end{bmatrix}$

$$\text{adj}(P) + Q = \begin{bmatrix} 5y & -x \\ -x^2 & 1 \end{bmatrix} + \begin{bmatrix} -7 & 3 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 5y-7 & -x+3 \\ -x^2+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 5y - 7 = 8 \Rightarrow y = 3$$

$$\Rightarrow \quad -x + 3 = 0 \Rightarrow x = 3$$

17. ***(d)*** Let number of boys and girls in the class be x and y, respectively.

$$\therefore \quad 62x + 54y = 60(x + y)$$

$$\Rightarrow \quad 2x = 6y$$

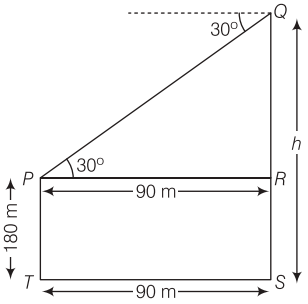
$$\Rightarrow \quad x = 3y$$

$\therefore$ Total number of students in the class

$$= x + y = 3y + y = 4y$$

$\therefore$ Percentage of boys in the class $= \frac{3y}{4y} \times 100\% = 75\%$

18. ***(a)*** In ΔPQR, $\tan 30^\circ = \frac{QR}{PR}$



$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h - 180}{90}$$

$$\therefore \quad h = (180 + 30\sqrt{3})\ \text{m}$$

19. ***(c)*** Consider
p : Two triangles are identical.
q : Two triangles are similar.
Given statement in symbolic form can be written as

$$p \Rightarrow q$$

$\therefore$ Its contrapositive will be $\sim q \Rightarrow \sim p$
In statement form,
$\sim q \Rightarrow \sim p$: If two triangles are not similar, then these are not identical.

20. ***(b)*** $\because \log_{10} 2, \log_{10}(5^x - 1)$ and $\log_{10}(5^x + 3)$ are in AP.

$$\therefore \quad 2\log_{10}(5^x - 1) = \log_{10} 2 + \log_{10}(5^x + 3)$$

$$\Rightarrow \quad (5^x - 1)^2 = 2(5^x + 3)$$

Let $5^x = y$

$$\therefore \quad (y-1)^2 = 2(y+3)$$

$\Rightarrow \quad y^2 - 4y - 5 = 0$

$\Rightarrow \quad (y - 5)(y + 1) = 0$

$\Rightarrow \quad y = 5 \text{ or } y = -1$

$\Rightarrow \quad 5^x = 5 \text{ or } 5^x = -1$

[but $5^x \neq -1$, because exponential function can never be negative]

$\therefore \quad 5^x = 5$

$\therefore \quad x = 1$

21. ***(b)*** On putting $x = 1$ in $(1 + 5x - 7x^2)^{3546}$, we can find the sum of coefficients.

$\therefore$ Sum of coefficient $= [1 + 5(1) - 7(1)^2]^{3546}$

$= [1 + 5 - 7]^{3546} = (-1)^{3546} = 1$

22. ***(d)*** Equation of plane containing the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ is

$a(x - 0) + b(y - 0) + c(z - 0) = 0 \quad \ldots(i)$

and $2a + 3b + 4c = 0 \quad \ldots(ii)$

Another equation of plane containing the other two lines is

$a_1(x - 0) + b_1(y - 0) + c_1(z - 0) = 0 \quad \ldots(iii)$

Also, $3a_1 + 4b_1 + 2c_1 = 0$ and $4a_1 + 2b_1 + 3c_1 = 0$

On solving, we get

$$\frac{a_1}{8} = \frac{b_1}{-1} = \frac{c}{-10}$$

$\therefore$ Eq. (iii) becomes

$8x - y - 10c = 0 \quad \ldots(iv)$

$\because$ The plane (i) is perpendicular to the plane (ii).

$\therefore \quad 8a - b - 10c = 0 \quad \ldots(v)$

On solving Eqs. (ii) and (v), we get

$$\frac{a}{-26} = \frac{b}{52} = \frac{c}{-26} \text{ or } \frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$$

From Eq. (i),

$x - 2y + z = 0$

23. ***(b)*** Let vector $\mathbf{p}$ makes angles α, β and γ with $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively.

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \ldots(i)$

We know, square of a vector is square of its modulus.

$\therefore \quad |\hat{\mathbf{i}} \times \mathbf{p}|^2 = (\hat{\mathbf{i}} \times \mathbf{p})^2 = (1 \cdot p\sin\alpha)^2 \cdot 1 = p^2\sin^2\alpha$

Similarly, we can write

$|\hat{\mathbf{j}} \times \mathbf{p}|^2 = p^2\sin^2\beta$ and $|\hat{\mathbf{k}} \times \mathbf{p}|^2 = p^2\sin^2\gamma$

$\therefore \; |\hat{\mathbf{i}} \times \mathbf{p}|^2 + |\hat{\mathbf{j}} \times \mathbf{p}|^2 + |\hat{\mathbf{k}} \times \mathbf{p}|^2$

$= p^2\sin^2\alpha + p^2\sin^2\beta + p^2\sin^2\gamma$

$= p^2(1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma)$

$= p^2[3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)]$

$= p^2[3 - 1] = 2p^2$

24. ***(a)*** A is the event that three defective machines are identified in first three tests out of eight machines.

$\therefore \quad P(A) = \frac{{}^3C_3}{{}^8C_3} = \frac{1}{56}$

25. ***(b)*** Given, number of flags = 7

$\therefore$ Total number of signals that can be formed = (Number of signals formed using one flag) + (Number of signals formed using two flags) + … + (Number of signals formed using 7 flags)

$= 7 + 42 + 210 + 840 + 2520 + 5040 + 5040$

$= 13699$

26. ***(c)*** $\because \int_3^6 \{4 - f(x)\}\,dx = 15$

$\Rightarrow \quad \int_3^6 4dx - \int_3^6 f(x)\,dx = 15$

$\Rightarrow \quad 4[x]_3^6 - \int_3^6 f(x)\,dx = 15$

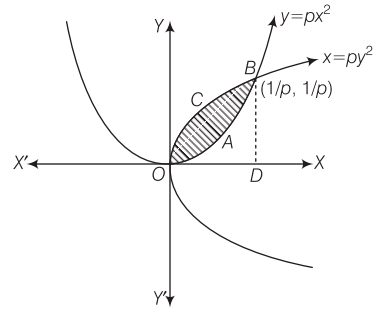
$\Rightarrow \quad 4[6 - 3] - \int_3^6 f(x)\,dx = 15$

$\Rightarrow \quad \int_3^6 f(x)\,dx = -3$

$\therefore \quad \int_{-4}^3 f(x)\,dx = \int_{-4}^6 f(x)\,dx - \int_3^6 f(x)\,dx$

$= 6 - (-3) = 9$

27. ***(d)*** The points of intersection between the curves $x = py^2$ and $y = px^2$ are $(0, 0)$ and $\left(\frac{1}{p}, \frac{1}{p}\right)$.



$\therefore$ Required area $OABCO$

= Area of $OCBDO$ – Area of $OABDO$

$\Rightarrow \quad \int_0^{1/p}\left(\sqrt{\frac{x}{p}} - px^2\right)dx = 3$

$\Rightarrow \quad \left[\frac{1}{\sqrt{p}} \cdot \frac{x^{3/2}}{3/2} - \frac{px^3}{3}\right]_0^{1/p} = 3$

$\Rightarrow \quad \frac{2}{3p^2} - \frac{1}{3p^2} = 3 \Rightarrow p = \frac{1}{3} \quad [\text{as } p > 0]$

28. ***(b)*** Given, integrating factor $= \sin^2 x$

We know,

integrating factor $= e^{\int P dx}$

$\therefore \quad e^{\int P dx} = \sin^2 x \Rightarrow \int P dx = \log(\sin^2 x)$

On differenting both sides, we get

$P = \frac{d}{dx}[\log(\sin^2 x)] = \frac{1}{\sin^2 x} \times 2\sin x \cdot \cos x = 2\cot x$

29. ***(b)*** We have, $f(x) = [\tan^2 x]$

$\because \lim_{x \to 0} f(x) = \lim_{x \to 0}[\tan^2 x] = 0$ and $f(0) = [\tan^2 0] = 0$

$\therefore f(x)$ is continuous at $x = 0$.

30. ***(a)*** Let $t = \tan\alpha$

$\therefore \quad \sin 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2t}{1 - t^2}$

$\cos 2\alpha = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} = \frac{1 - t^2}{1 + t^2}$

$\therefore \quad \tan^{-1}\frac{3t}{4 + t^2} + \tan^{-1}\frac{t}{4} = \tan^{-1}t\frac{(t^2 + 16)}{(t^2 + 16)}$

$= \tan^{-1}t = \tan^{-1}(\tan\alpha) = \alpha$

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JEE MAIN

(PAPER TWO SCALE UP)

FULL TEST 6

(COMPLETE SYLLABUS) WITH ANSWER KEY

Mock Questions from complete syllabus to enhance Your Problem Solving Skills

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. A and B participate in a tournament of 'best of 7 games'. It is equally likely that either A wins or B wins or the games ends in a draw. Then, the probability that A wins the tournament, is

(a) $\frac{271}{729}$ *(b)* $\frac{399}{422}$ *(c)* $\frac{299}{729}$ *(d)* $\frac{199}{729}$

2. The value of the $\sum_{n=0}^{\infty} \frac{2n+3}{3^n}$ is equal to

(a) 5 *(b)* 6 *(c)* 7 *(d)* 2

3. If a pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ is such that each pair bisects the angle between the other pair. Then, pq is equal to

(a) 1 *(b)* 2 *(c)* – 2 *(d)* – 1

4. The equation of circle which touches the X-axis and the line $4x - 3y + 4 = 0$. Its centre lying in the third quadrant and lies on the line $x - y - 1 = 0$, is

(a) $9(x^2 + y^2) + 6x + 24y + 1 = 0$
(b) $9(x^2 + y^2) - 6x - 24y - 1 = 0$
(c) $4(x^2 + y^2) + 6x + 27y + 1 = 0$
(d) None of the above

5. The sum of

$$\frac{3}{1\times 2}\times\frac{1}{2} + \frac{4}{2\times 3}\times\left(\frac{1}{2}\right)^2 + \frac{5}{3\times 4}(1/2)^3 + \ldots$$

to n terms, is

(a) $1 + \frac{1}{(n+1)2^n}$ *(b)* $1 - \frac{1}{(n+1)3^n}$

(c) $1 + \frac{1}{(2n+1)}$ *(d)* $1 - \frac{1}{(n+1)2^n}$

6. The extremities of a diagonal of rectangle are $(-4, 4)$ and $(6, -1)$. A circle circumscribes the rectangle and cuts an intercept AB on the Y-axis. Then, the area of the triangle formed by AB and tangents to the circle at A and B, is

(a) $\frac{1331}{8}$ sq units *(b)* $\frac{1241}{7}$ sq units

(c) $\frac{1301}{4}$ sq units *(d)* None of these

7. If the normal at $(18, 12)$ to the parabola $y^2 = 8x$ cuts it again at Q. Then, the value of $9PQ$ is

(a) $70\sqrt{10}$ *(b)* $80\sqrt{10}$ *(c)* $90\sqrt{7}$ *(d)* $40\sqrt{3}$

8. The value of $\int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)}\,dx$ is

(a) $\frac{1}{3}\left(\log 6 - \frac{1}{4}\right)$ *(b)* $\frac{1}{2}\left(\log 6 - \frac{1}{5}\right)$

(c) $\frac{1}{5}\left(\log 5 - \frac{1}{3}\right)$ *(d)* $\frac{1}{3}\left(\log 2 - \frac{1}{3}\right)$

9. The value of $\int \frac{x^2\left(1 - \frac{1}{x^2}\right)dx}{x^2\left[\sqrt{\left(x + \frac{1}{x} + \alpha\right)}\sqrt{\left(x + \frac{1}{x} + \beta\right)}\right]}$ is

(a) $2\log\left(\frac{\sqrt{x^2 + 2x + 1} - \sqrt{x^2 + \beta x + 1}}{\sqrt{x}}\right)^2 + C$

(b) $\log\left(\frac{\sqrt{2x^2 + x + 1} + \sqrt{\beta x + 1}}{\sqrt{x}}\right)^2 + C$

(c) $3\log\left(\frac{\sqrt{x^2+2x+1}+\sqrt{x^2+\beta x+1}}{\sqrt{x}}\right)^2+C$

(d) None of the above

10. A normal inclined at 45° to the X-axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is drawn. It meets the major and minor axes in P and Q, respectively. If C is the centre of the ellipse, then the area of ΔCPQ is

(a) $\frac{(a^2-b^2)^2}{2(a^2+b^2)}$ sq unit (b) $\frac{(a^2+b^2)^2}{2(a^2-b^2)}$ sq unit

(c) $\frac{(3a^2-b^2)}{2(a^2+b^2)}$ sq unit (d) None of these

11. The square root of $x+i\sqrt{x^4+x^2+1}$ is

(a) $\pm\frac{1}{\sqrt{2}}[\sqrt{x^2+x+1}+i\sqrt{x^2-x+1}]$

(b) $\pm\frac{1}{\sqrt{3}}[\sqrt{x^2+1}+i\sqrt{x^2-x+1}]$

(c) $\pm\frac{1}{\sqrt{2}}[\sqrt{x^2-x+1}+i\sqrt{x^2+x-1}]$

(d) $\pm\frac{1}{\sqrt{2}}[\sqrt{x^2-x-1}+\sqrt{x^2+2x+1}]$

12. If α,β are the roots of the equation $x+1=\lambda x(1-\lambda x)$ and λ_1,λ_2 are the two values of λ determine from the equation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\pi-2$. Then, the value of $\frac{\lambda_1^2}{\lambda_2^2}+\frac{\lambda_2^2}{\lambda_1^2}+2$ is equal to

(a) $3\left(\frac{\pi+1}{\pi-1}\right)^2$ (b) $3\left(\frac{2\pi+1}{2\pi-1}\right)^2$ (c) $4\left(\frac{\pi+1}{\pi-1}\right)^2$ (d) $4\left(\frac{\pi-1}{\pi+1}\right)^2$

13. If $\begin{vmatrix} x^n & x^{n+2} & x^{n+4} \\ y^n & y^{n+2} & y^{n+4} \\ z^n & z^{n+2} & z^{n+4} \end{vmatrix}$

$=\left(\frac{1}{y^2}-\frac{1}{x^2}\right)\left(\frac{1}{z^2}-\frac{1}{y^2}\right)\left(\frac{1}{x^2}-\frac{1}{z^2}\right)$, then $-n$ is equal to

(a) 3 (b) 4 (c) -2 (d) -4

14. If $A=\begin{bmatrix}0&1&2\\1&2&3\\3&a&1\end{bmatrix}$ and $A^{-1}=\begin{bmatrix}1/2&1/2&1/2\\-4&3&c\\5/2&-3/2&1/2\end{bmatrix}$, then the values of a and c are equal to

(a) 1, 1 (b) 1, -1 (c) 1, 2 (d) -1, 1

15. A question paper consists of two sections having respectively 3 and 4 questions. The following note is given on the paper "It is not necessary to attempt all the questions. One question from each section is compulsory". The candidate can select the question in way is equal to

(a) 104 (b) 102 (c) 103 (d) 105

16. If the 3rd, 4th, 5th and 6th terms in the expansion of $(x+\alpha)^n$ be respectively a, b, c and d, then the value of $\frac{b^2-ac}{c^2-bd}$ is

(a) $\frac{5a}{3c}$ (b) $\frac{2a}{3c}$ (c) $\frac{2a}{3b}$ (d) $\frac{5a}{3b}$

17. A hemispherical bowl of radius a is filled with water upto a depth h. The volume of water in the bowl is

(a) $\pi h^2\left(a+\frac{h}{3}\right)$ (b) $\pi h^2\left(a-\frac{h}{3}\right)$

(c) $\pi h^3\left(a+\frac{2h}{3}\right)$ (d) None of these

18. An aeroplane is flying horizontally at a height of 2/3 km with velocity of 15 km/h. Then, the rate at which it is receding from a fixed point on the ground which it passed over 2 min ago, is

(a) 8 km/h (b) 10 km/h (c) 12 km/h (d) 9 km/h

19. If $f(x)=\begin{cases}\frac{8^x-4^x-2^x+1}{x^2}, & x>0\\ e^x\sin x+\pi x+\lambda\log 4, & x\le 0\end{cases}$ is continuous at $x=0$. Then, the value of λ is

(a) $4\log_e 2$ (b) $2\log_e 2$

(c) $\log_e 2$ (d) None of these

20. A line with positive direction cosines passes through the point $P(2,-1,2)$ and makes equal angles with the coordinate axes. The line meet the plane $2x+y+z=9$ at point Q. The length of the line segment PQ is

(a) $\sqrt{3}$ (b) $\sqrt{5}$ (c) $\sqrt{2}$ (d) $\sqrt{3/2}$

21. If $\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{c}}$ are unit vectors such that $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=0=\hat{\mathbf{a}}\cdot\hat{\mathbf{c}}$ and the angle between $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ is $\frac{\pi}{3}$, then the value of $|\hat{\mathbf{a}}\times\hat{\mathbf{b}}-\hat{\mathbf{a}}\times\hat{\mathbf{c}}|$ is

(a) -1 (b) 2 (c) 3 (d) 1

22. If the planes $x-cy-bz=0$, $cx-y+az=0$ and $bx+ay-z=0$ pass through a straight line, then the value of $a^2+b^2+c^2+2abc$ is

(a) 2 (b) -2 (c) 1 (d) -1

23. The sum of the square deviations for 10 observations taken from their mean 50 is 250. The coefficient of variation is

(a) 10% (b) 40%

(c) 90% (d) None of these

24. $\sim(p\vee q)\vee(\sim p\wedge q)$ is logically equivalent to

(a) $\sim p$ (b) p

(c) q (d) $\sim q$

25. The value of $\lim_{x\to 0}\frac{\sin(3x+a)-3\sin(2x+a)+3\sin(x+a)-\sin a}{x^3}$ is

(a) $\cos a$ (b) $\sin a$ (c) $-\cos a$ (d) $-\sin a$

To be Continued at Page 67

CONIC SECTION

A collection of best questions asked in JEE to revise your concepts

Single Option Correct Type

1. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is [IIT JEE 2012]

(a) $20(x^2 + y^2) - 36x + 45y = 0$

(b) $20(x^2 + y^2) + 36x - 45y = 0$

(c) $36(x^2 + y^2) - 20x + 45y = 0$

(d) $36(x^2 + y^2) + 20x - 45y = 0$

2. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is [IIT JEE 2007]

(a) 3 (b) 2 (c) $\frac{3}{2}$ (d) 1

3. The ΔPQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (– 4, 3) respectively, then $\angle QPR$ is equal to [IIT JEE 2000]

(a) $\pi / 2$ (b) $\pi / 3$ (c) $\pi / 4$ (d) $\pi / 6$

4. The centre of circle inscribed in square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$, is [IIT JEE 2003]

(a) (4, 7) (b) (7, 4) (c) (9, 4) (d) (4, 9)

5. If axis of a parabola is $y = x$ and vertex and focus are at a distance $\sqrt{2}$ and $2\sqrt{2}$ respectively from the origin. Then, equation of the parabola is [IIT JEE 2006]

(a) $(x - y)^2 = 8(x + y - 2)$ (b) $(x + y)^2 = 2(x + y - 2)$

(c) $(x - y)^2 = 4(x + y - 2)$ (d) $(x + y)^2 = 2(x - y + 2)$

6. If the focal chord to $y^2 = 16x$ is tangent to $(x-6)^2 + y^2 = 2$, then the possible values of the slope of this chord are [IIT 2003]

(a) {–1 , 1 } (b) {–2, 2}

(c) {–2, 1 / 2} (d) {2, – 1 / 2}

7. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the X-axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latusrectum of the given ellipse at the points [IIT JEE 2009]

(a) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$

(c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

8. The area (in sq units) of the quadrilateral formed by the tangents at the end points of the latusrectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is [JEE Mains 2015]

(a) $\frac{27}{4}$ (b) 18 (c) $\frac{27}{2}$ (d) 27

9. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R, whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is [IIT JEE 2012]

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

10. Consider a branch of the hyperbola

$$x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$$

with vertex at the point A. Let B be one of the end points of its latusrectum. If C is the focus of the hyperbola nearest to the point A, then the area of the ΔABC is [IIT JEE 2008]

(a) $(1 - \sqrt{2/3})$ sq unit (b) $(\sqrt{3/2} - 1)$ sq unit

(c) $(1 + \sqrt{2/3})$ sq unit (d) $(\sqrt{3/2} + 1)$ sq unit

Single Integer Type

11. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the centre, angles of π / k and $2\pi / k$, where $k > 0$, then the value of $[k]$ is ....., where $[k]$ denotes the largest integer less than or equal to k. [IIT JEE 2010]

12. Let S be the focus of the parabola $y^2 = 8$ and PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of ΔQPS is ..... . [IIT JEE 2012]

13. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the X-axis, then the eccentricity of the hyperbola is ..... . [IIT JEE 2010]

14. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of ΔPQR, $\Delta_1 = \max_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min_{1/2 \le h \le 1} \Delta(h)$, then $8/\sqrt{5}\,\Delta_1 - 8\Delta_2$ is equal to ..... . [IIT JEE 2013]

More Than One Option Correct Type

15. Circle(s) touching X-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on Y-axis, is/are [JEE Advanced 2013]

(a) $x^2 + y^2 - 6x + 8y + 9 = 0$

(b) $x^2 + y^2 - 6x + 7y + 9 = 0$

(c) $x^2 + y^2 - 6x - 8y + 9 = 0$

(d) $x^2 + y^2 - 6x - 7y + 9 = 0$

16. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of ΔPTN is a parabola, whose [JEE Advanced 2009]

(a) vertex is $\left(\frac{2a}{3}, 0\right)$ (b) directrix is $x = 0$

(c) latusrectum is $\frac{2a}{3}$ (d) focus is $(a, 0)$

17. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be [IIT JEE 2010]

(a) $-\frac{1}{r}$ (b) $\frac{1}{r}$ (c) $\frac{2}{r}$ (d) $-\frac{2}{r}$

18. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line $2x - y = 1$. The points of contacts of the tangents on the hyperbola are [IIT JEE 2012]

(a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(c) $(3\sqrt{3}, -2\sqrt{2})$ (d) $(-3\sqrt{3}, 2\sqrt{2})$

Linked Comprehension Type

Directions (Q. Nos. 19-20) Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$. [JEE Advanced 2013]

19. Length of chord PQ is

(a) $7a$ (b) $5a$ (c) $2a$ (d) $3a$

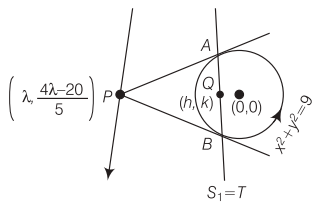
20. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta$ is equal to

(a) $\frac{2}{3}\sqrt{7}$ (b) $\frac{-2}{3}\sqrt{7}$ (c) $\frac{2}{3}\sqrt{5}$ (d) $\frac{-2}{3}\sqrt{5}$

Answers with Explanation

1. **(a)** Here, equation of chord of contact w.r.t. P is

$$x\lambda + y\left(\frac{4\lambda-20}{5}\right) = 9 \Rightarrow 5\lambda x + (4\lambda - 20)y = 45 \quad \ldots(i)$$



and equation of chord bisected at the point $Q(h,k)$ is

$$xh + yk - 9 = h^2 + k^2 - 9 \Rightarrow xh + ky = h^2 + k^2 \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{5\lambda}{h} = \frac{4\lambda-20}{k} = \frac{45}{h^2+k^2}$$

$$\therefore \quad \lambda = \frac{20h}{4h-5k} \text{ and } \lambda = \frac{9h}{h^2+k^2}$$

$$\Rightarrow \quad \frac{20h}{4h-5k} = \frac{9h}{h^2+k^2} \Rightarrow 20(h^2+k^2) = 9(4h-5k)$$

$$\Rightarrow \quad 20(x^2+y^2) = 36x - 45y$$

2. **(b)** Let $CD = \alpha$. Then, $AB = 2\alpha$

The equation of BC is $y = \frac{-2r}{\alpha}(x - 2\alpha)$

$$\Rightarrow \quad 2rx + \alpha y - 4r\alpha = 0$$

It touches the circle $(x - r)^2 + (y - r)^2 = r^2$.

$$\therefore \quad \left|\frac{2r^2 + \alpha r - 4\alpha r}{\sqrt{4r^2+\alpha^2}}\right| = r$$

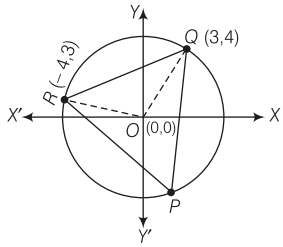
$$\Rightarrow \quad (2r^2 - 3\alpha r)^2 = r^2(\alpha^2 + 4r^2) \Rightarrow \alpha = \frac{3r}{2}$$

Now, area of quadrilateral $ABCD = 18$

$$\Rightarrow \quad \frac{1}{2}(AB + CD) \times 2r = 18 \Rightarrow 3\alpha r = 18$$

$$\Rightarrow \quad 3r^2 = 12 \Rightarrow r = 2$$

3. **(c)** Let O be the point at centre and P be the point at circumference. Therefore, $\angle QOR$ is double the $\angle QPR$. So, it is sufficient to find $\angle QOR$.



Now, slope of $OQ, m_1 = 4/3$, and slope of OR, $m_2 = -3/4$

Here, $m_1 m_2 = -1$

Therefore, $\angle QOR = \pi/2$

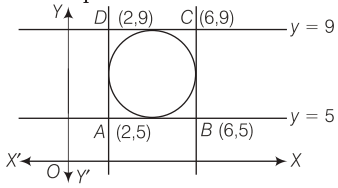
which implies that $\angle QPR = \pi/4$

4. **(a)** Given, circle is inscribed in square formed by the lines

$$x^2 - 8x + 12 = 0 \text{ and } y^2 - 14y + 45 = 0$$

$$\Rightarrow \quad x = 6 \text{ and } x = 2,\ y = 5 \text{ and } y = 9$$

which could be plotted as



Here, $ABCD$ clearly forms a square.

$\therefore$ Centre of inscribed circle

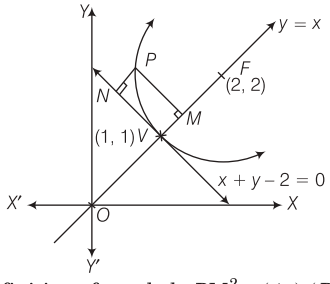
= Point of intersection of diagonals

= Mid-point of AC or $BD = \left(\frac{2+6}{2}, \frac{5+9}{2}\right) = (4, 7)$

5. ***(a)*** Since, distance of vertex from origin is $\sqrt{2}$ and focus is $2\sqrt{2}$.

$\therefore$ $V(1, 1)$ and $F(2, 2)$ (i.e. lying on $y = x$)

where, length of latusrectum $= 4a = 4\sqrt{2}$ $[\because a = \sqrt{2}]$



$\therefore$ By definition of parabola $PM^2 = (4a)(PN)$

where, PN is length of perpendicular upon $x + y - 2 = 0$, i.e. tangent at vertex

$$\Rightarrow \quad \frac{(x-y)^2}{2} = 4\sqrt{2}\left(\frac{x+y-2}{\sqrt{2}}\right)$$

$$\Rightarrow \quad (x-y)^2 = 8(x+y-2)$$

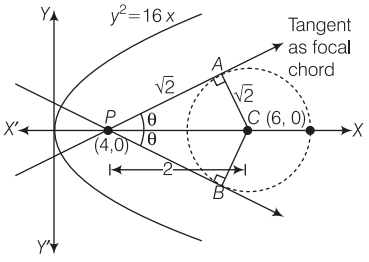
6. ***(a)*** Here, the focal chord of $y^2 = 16x$ is tangent to circle $(x-6)^2 + y^2 = 2$.

$\Rightarrow$ Focus of parabola is $(a, 0)$ i.e. $(4, 0)$

Now, tangents are drawn from $(4, 0)$ to $(x-6)^2 + y^2 = 2$.

Since, PA is tangent to circle.

$\therefore$ $\tan\theta$ = slope of tangent $= \frac{AC}{AP} = \frac{\sqrt{2}}{\sqrt{2}} = 1$, or $\frac{BC}{BP} = -1$



$\therefore$ Slope of focal chord as tangent to circle $= \pm 1$

7. ***(c)*** Given, $\frac{x^2}{16} + \frac{y^2}{4} = 1$

Here, $a = 4, b = 2$

Equation of normal $4x \sec\theta - 2y \operatorname{cosec}\theta = 12$

$$M\left(\frac{7\cos\theta}{2}, \sin\theta\right) = (h, k) \qquad \text{[say]}$$

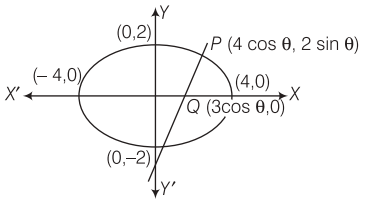
$\therefore$ $h = \frac{7\cos\theta}{2} \Rightarrow = \frac{2h}{7} = \cos\theta$ ...(i)

and $k = \sin\theta$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$\frac{4h^2}{49} + k^2 = 1 \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

Hence, locus is $\frac{4x^2}{49} + y^2 = 1$ ...(iii)



For given ellipse, $e^2 = 1 - \frac{4}{16} = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$

$\therefore$ $x = \pm 4 \times \frac{\sqrt{3}}{2} = \pm 2\sqrt{3}$ $[\because x = \pm ae]$ ...(iv)

On solving Eqs. (iii) and (iv), we get

$$\frac{4}{49} \times 12 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 - \frac{48}{49} = \frac{1}{49}, \quad y = \pm\frac{1}{7}$$

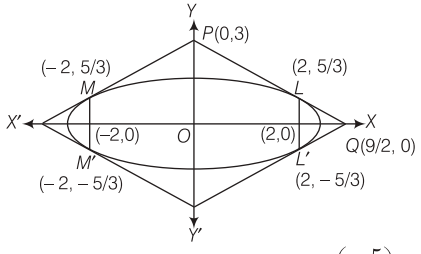
$\therefore$ Required points are $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$.

8. ***(d)*** Given equation of ellipse is $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$\therefore$ $a^2 = 9, b^2 = 5 \Rightarrow a = 3, b = \sqrt{5}$

Now, $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

Foci $= (\pm ae, 0) = (\pm 2, 0)$ and $\frac{b^2}{a} = \frac{5}{3}$



$\therefore$ Extremities of one of latusrectum are $\left(2, \frac{5}{3}\right)$ and $\left(2, \frac{-5}{3}\right)$

$\therefore$ Equation of tangent at $\left(2, \frac{5}{3}\right)$ is

$$\frac{x(2)}{9} + \frac{y(5/3)}{5} = 1 \quad \Rightarrow \quad 2x + 3y = 9$$

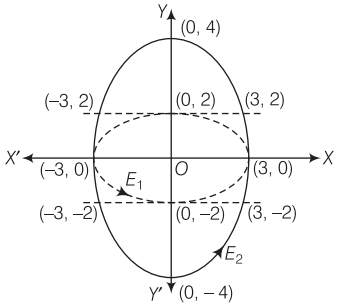
Since, Eq. (ii) intersects X and Y-axes at $\left(\frac{9}{2}, 0\right)$ and $(0, 3)$, respectively.

$\therefore$ Area of quadrilateral $= 4 \times$ Area of ΔPOQ

$$= 4 \times \left(\frac{1}{2} \times \frac{9}{2} \times 3\right) = 27 \text{ sq units}$$

9. ***(c)*** Let the equation of an ellipse E_2 be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a < b \text{ and } b = 4.$$



Also, it passes through (3, 2).

$\Rightarrow \frac{9}{a^2}+\frac{4}{b^2}=1 \Rightarrow \frac{9}{a^2}+\frac{1}{4}=1 \Rightarrow a^2=12 \quad [\because b=4]$

Eccentricity of E_2, $e^2=1-\frac{a^2}{b^2}=1-\frac{12}{16}=\frac{1}{4} \quad [\because a<b]$

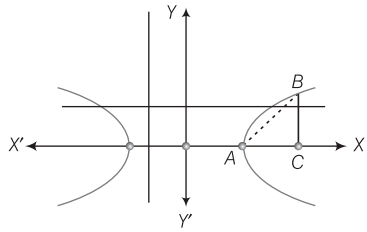
$\therefore \quad e=\frac{1}{2}$

10. ***(b)*** Given equation can be rewritten as

$$\frac{(x-\sqrt{2})^2}{4}-\frac{(y+\sqrt{2})^2}{2}=1$$

$\Rightarrow \quad e=\sqrt{1+\frac{2}{4}}=\sqrt{\frac{3}{2}}$

For point $A(x, y), x-\sqrt{2}=2 \Rightarrow x=2+\sqrt{2}$



For point $C(x, y), x-\sqrt{2}=ae=\sqrt{6} \Rightarrow x=\sqrt{6}+\sqrt{2}$

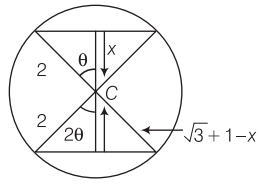
Now, $AC=\sqrt{6}+\sqrt{2}-2-\sqrt{2}=\sqrt{6}-2$

and $BC=\frac{b^2}{a}=\frac{2}{2}=1$

$\therefore$ Area of $\Delta ABC=\frac{1}{2}\times(\sqrt{6}-2)\times 1=\sqrt{\frac{3}{2}}-1$ sq unit

11. ***(3)*** Let $\theta=\frac{\pi}{2k} \Rightarrow \cos\theta=\frac{x}{2}$

$\Rightarrow \quad \cos 2\theta=\frac{\sqrt{3}+1-x}{2}$



$\Rightarrow \quad 2\cos^2\theta-1=\frac{\sqrt{3}+1-x}{2}$

$\Rightarrow \quad 2\left(\frac{x^2}{4}\right)-1=\frac{\sqrt{3}+1-x}{2}$

$\Rightarrow \quad x^2+x-3-\sqrt{3}=0$

$\Rightarrow \quad x=\frac{-1\pm\sqrt{1+12+4\sqrt{3}}}{2}$

$=\frac{-1\pm\sqrt{13+4\sqrt{3}}}{2}$

$=\frac{-1+2\sqrt{3}+1}{2}=\sqrt{3}$

$\therefore \quad \cos\theta=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{6}$

$\therefore$ Required angle $=\frac{\pi}{k}=2\theta=\frac{\pi}{3} \Rightarrow k=3$

12. ***(4)*** The equation of parabola in parametric form is $x=2t^2, y=4t$

Solving it with the circle, we get

$4t^4+16t^2-4t^2-16t=0$

$\Rightarrow \quad t^4+3t^2-4t=0$

$\Rightarrow \quad t(t-1)(t^2+t+4)=0$

$\Rightarrow \quad t=0, 1$

So, the points P and Q are (0, 0) and (2, 4) which are also diametrically opposite points on the circle.

Also, the focus is $S \equiv (2, 0)$

$\therefore$ Area of $\Delta PQS=\frac{1}{2}\times PS\times QS=\frac{1}{2}\times 2\times 4=4$

13. ***(2)*** On substituting $\left(\frac{a}{e}, 0\right)$ in $y=-2x+1$, we get

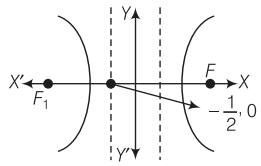
$$0=-\frac{2a}{e}+1 \quad \Rightarrow \frac{a}{e}=\frac{1}{2}$$

Also, $y=-2x+1$ is tangent to hyperbola.

$\therefore \quad 1=4a^2-b^2 \Rightarrow \frac{1}{a^2}=4-(e^2-1) \Rightarrow \frac{4}{e^2}=5-e^2$

$\Rightarrow e^4-5e^2+4=0 \Rightarrow (e^2-4)(e^2-1)=0$

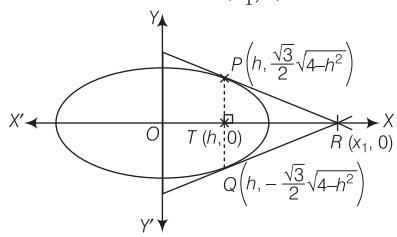
$\Rightarrow \quad e=2, e=1$



$e=1$ gives the conic as parabola. But conic is given as hyperbola, hence $e=2$.

14. ***(9)*** Given, $\frac{x^2}{4}+\frac{y^2}{3}=1 \Rightarrow y=\frac{\sqrt{3}}{2}\sqrt{4-h^2}$ at $x=h$

Let the coordinates of R be $(x_1, 0)$.



PQ is chord of contact.

$\therefore \quad \frac{xx_1}{4}=1 \Rightarrow x=\frac{4}{x_1}$

which is equation of PQ, $x=h$

$\therefore \quad \frac{4}{x_1}=h \Rightarrow x_1=\frac{4}{h}$

$\Delta(h)=$ Area of $\Delta PQR=\frac{1}{2}\times PQ\times RT$

$=\frac{1}{2}\times\frac{2\sqrt{3}}{2}\sqrt{4-h^2}\times(x_1-h)=\frac{\sqrt{3}}{2h}(4-h^2)^{3/2}$

$\Rightarrow \quad \Delta'(h)=\frac{-\sqrt{3}(4+2h^2)}{2h^2}\sqrt{4-h^2}$

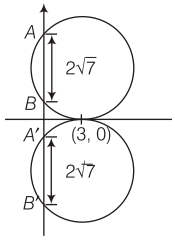
which is always decreasing.

So, $\Delta_1=$ maximum of $\Delta(h)=\frac{45\sqrt{5}}{8}$ at $h=\frac{1}{2}$

Δ_2 = minimum of $\Delta(h) = \frac{9}{2}$ at $h = 1$

So, $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = \frac{8}{\sqrt{5}} \times \frac{45\sqrt{5}}{8} - 8 \times \frac{9}{2} = 45 - 36 = 9$

15. ***(a, c)*** Here, $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through (3, 0).



$$\Rightarrow \quad 9 + 6g + c = 0 \qquad \ldots(i)$$

$$g^2 = c \qquad \ldots(ii)$$

and $2\sqrt{f^2 - c} = 2\sqrt{7}$

$$f^2 - c = 7 \qquad \ldots(iii)$$

From Eqs. (i) and (ii), we get

$$g^2 + 6g + 9 = 0$$

$$\Rightarrow \quad (g + 3)^2 = 0$$

$$\Rightarrow \quad g = -3 \text{ and } c = 9$$

$$\therefore \quad f^2 = 16 \Rightarrow f = \pm 4$$

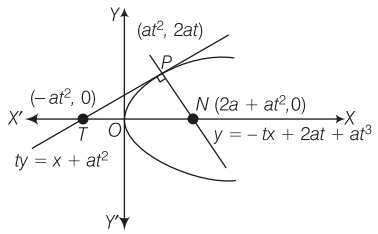
$$\therefore \quad x^2 + y^2 - 6x \pm 8y + 9 = 0$$

16. ***(a, d)*** Equations of tangent and normal at point $P(at^2, 2at)$ are

$$ty = x + at^2 \quad \text{and} \quad y = -tx + 2at + at^3$$

Let centroid of ΔPTN be $R(h, k)$.

$$\therefore \quad h = \frac{at^2 + (-at^2) + 2a + at^2}{3}$$



and $k = \frac{2at}{3} \Rightarrow 3h = 2a + a \cdot \left(\frac{3k}{2a}\right)^2$

$$\Rightarrow \quad 3h = 2a + \frac{9k^2}{4a}$$

$$\Rightarrow \quad 9k^2 = 4a(3h - 2a)$$

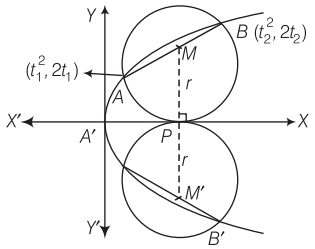
$\therefore$ Locus of centroid is $y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

$\therefore$ Vertex $\left(\frac{2a}{3}, 0\right)$; directrix $x - \frac{2a}{3} = -\frac{a}{3} \Rightarrow x = \frac{a}{3}$

and latusrectum $= \frac{4a}{3}$

$\therefore$ Focus $\equiv \left(\frac{a}{3} + \frac{2a}{3}, 0\right) \equiv (a, 0)$

17. ***(c, d)*** Here, coordinate $M = \left(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$ i.e. mid-point of chord AB.



$$MP = t_1 + t_2 = r \qquad \ldots(i)$$

Also, $m_{AB} = \frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = \frac{2}{t_2 + t_1}$ [when AB is chord]

$$\Rightarrow \quad m_{AB} = \frac{2}{r} \qquad \text{[from Eq. (i)]}$$

Also, $m_{A'B'} = -\frac{2}{r}$ [when $A'B'$ is chord]

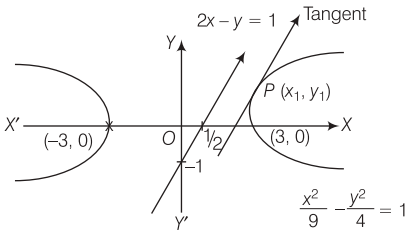
18. ***(a, b)*** Equation of tangent, parallel to $y = 2x - 1$ is

$$y = 2x \pm \sqrt{9(4) - 4}$$

$$\therefore \quad y = 2x \pm \sqrt{32} \qquad \ldots(i)$$

The equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{9} - \frac{yy_1}{4} = 1 \qquad \ldots(ii)$$



From Eqs. (i) and (ii), we get

$$\frac{\frac{2}{x_1}}{9} = \frac{-1}{\frac{-y_1}{4}} = \frac{\pm\sqrt{32}}{1} \Rightarrow x_1 = -\frac{9}{2\sqrt{2}} \text{ and } y_1 = -\frac{1}{\sqrt{2}}$$

or $x_1 = \frac{9}{2\sqrt{2}}$ and $y_1 = \frac{1}{\sqrt{2}}$

19. ***(b)*** Let $P(at^2, 2at)$, $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ as PQ is focal chord.

Point of intersection of tangents at P and Q is $\left(-a, a\left(t - \frac{1}{t}\right)\right)$.

As this point lies on $y = 2x + a$

$$\Rightarrow \quad a\left(t - \frac{1}{t}\right) = -2a + a \Rightarrow t - \frac{1}{t} = -1$$

$$\therefore \quad \left(t + \frac{1}{t}\right)^2 = \left(t - \frac{1}{t}\right)^2 + 4 = 1 + 4 = 5$$

Length of focal chord $= a\left(t + \frac{1}{t}\right)^2 = 5a$

20. ***(d)*** Angle made by chord PQ at vertex (0, 0) is given by

$$\tan\theta = \frac{\left(\frac{2}{t} + 2t\right)}{1 - 4} = \frac{2\left(t + \frac{1}{t}\right)}{-3} = \frac{-2}{3}\sqrt{5}$$

Learn Through Problems

APPLICATION OF DERIVATIVE

Section targetted to provide stepwise learning & practice through selective problems

1 The equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$, is

(a) $x + y = 2$ (b) $x + y = 1$

(c) $x - y = 1$ (d) None of these

(b) ***Step I*** Find the value of y-coordinate, if x-coordinate is given or *vice-versa*.

We have, $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ ...(i)

When $x = 0$, we have

$y = (1 + 0)^y + \sin^{-1}(\sin^2 0) \Rightarrow y = 1$

Step II Differentiate the given equation of the curve w.r.t. x and find the slope of the tangent and normal.

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = (1 + x)^y \left\{\frac{dy}{dx} \log (1 + x) + \frac{y}{1 + x}\right\} + \frac{\sin 2x}{\sqrt{1 - \sin^4 x}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = 1 = \text{Slope of tangent}$$

$$\therefore \text{ Slope of normal} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(0,1)}} = -1$$

Step III Using point-slope form of the equation of a line i.e. $y - y_1 = m(x - x_1)$, find the equation of tangent or normal. So, the equation of the normal at $(0, 1)$ is

$$y - 1 = -1 (x - 0)$$

$$\Rightarrow \quad x + y = 1$$

TRY THESE

1. The equation of the tangent to the curve $y = 1 - e^{x/2}$ at the point of intersection with the Y-axis, is

(a) $x + 2y = 0$ (b) $2x + y = 0$

(c) $x - y = 2$ (d) None of these

2. The equation of the tangent to the curve $y = (2x - 1)e^{2(1-x)}$ at the point of its maximum, is

(a) $y - 1 = 0$ (b) $x - 1 = 0$

(c) $x + y - 1 = 0$ (d) $x - y + 1 = 0$

3. The equation of the normal to the curve $y = e^{-2|x|}$ at the point, where the curve cuts the line $x = -\frac{1}{2}$, is

(a) $2e(ex + 2y) = 4 - e^2$ (b) $2e(ex - 2y) = e^2 - 4$

(c) $2e(ey - 2x) = e^2 - 4$ (d) $2e(ey + 2x) = e^2 - 4$

1. (a) **2.** (a) **3.** (a)

2 The function $f(x) = 2 \log (x - 2) - x^2 + 4x + 1$ increases on the interval

(a) $(1, 2)$ (b) $(2, 3)$ (c) $(1, 3)$ (d) $(2, 4)$

(b) ***Step I*** Find the domain of the given function.

Clearly, $f(x)$ is defined for all $x > 2$.

Step II Differentiate the given function w.r.t. x.

We have, $f(x) = 2 \log(x - 2) - x^2 + 4x + 1$

$$\Rightarrow \quad f'(x) = \frac{2}{x - 2} - 2x + 4$$

$$\Rightarrow \quad f'(x) = 2\left\{\frac{1 - (x - 2)^2}{x - 2}\right\}$$

$$\Rightarrow \quad f'(x) = \frac{-2(x - 1)(x - 3)}{(x - 2)}$$

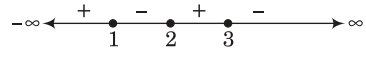
$$\Rightarrow \quad f'(x) = \frac{-2(x - 1)(x - 2)(x - 3)}{(x - 2)^2}$$

Step III Solve the inequality $f'(x) > 0$ or $f'(x) < 0$.

Clearly,

$f'(x) > 0$ for $x \in (-\infty, 1) \cup (2, 3)$

and $f'(x) < 0$ for $x \in (1, 2) \cup (3, \infty)$



Step IV Function $f(x)$ will be increasing for which $f'(x) > 0$ and decreasing for which $f'(x) < 0$ within its domain.

Thus, $f(x)$ is increasing on $(2, 3)$.

TRY THESE

1. $f(x) = |x \log_e x|$, $x > 0$ is monotonically decreasing in

(a) (e, ∞) (b) $(0, 1/e)$ (c) $[1/e, 1]$ (d) $(1, e)$

2. Let $f(x) = \log(\sin x + \cos x)$, $x \in \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$. Then, f is strictly increasing in the interval

(a) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (b) $\left(0, \frac{3\pi}{8}\right)$ (c) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

3. The interval in which the function $f(x) = \int_0^x \left(\frac{t}{t + 2} - \frac{1}{t}\right) dt$ will be non-increasing, is

(a) $(-2, -1] \cup (0, 3]$ (b) $(-2, -1] \cup [0, 3]$

(c) $(-2, -1] \cup [0, 2]$ (d) $(-2, -1] \cup (0, 2]$

1. (c) **2.** (a) **3.** (d)

3 On the interval $[0, 1]$, the function $x^{25}(1-x)^{75}$ takes its maximum value at the point

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

(b) ***Step I*** Put $y = f(x)$

$\therefore \quad y = x^{25}(1-x)^{75}$

Step II Find $\frac{dy}{dx}$.

$\therefore \quad \frac{dy}{dx} = 25(x)^{24}(1-x)^{75} + x^{25}(75)(1-x)^{74}(-1)$

$= 25x^{24}(1-x)^{74}(1-x-3x)$

$= 25x^{24}(1-x)^{74}(1-4x)$

Step III Put $\frac{dy}{dx} = 0$ and solve this equation for x.

Let $c_1, c_2, c_3, \ldots, c_n$ be the roots of this equation.

Now, $\frac{dy}{dx} = 0$

$\Rightarrow \quad 25x^{24}(1-x)^{74}(1-4x) = 0 \Rightarrow x = 0, 1, \frac{1}{4}$

Step IV Consider $x = c_1$. If $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through c_1, then the function attains a local maximum at $x = c_1$.

If $\frac{dy}{dx}$ changes its sign from negative to positive as x increases through c_1, then the function attains a local minimum at $x = c_1$.

If $\frac{dy}{dx}$ does not change sign as x increases through c_1, then $x = c_1$ is neither a point of local maximum nor a point of local minimum. In this case, $x = c_1$ is a point of inflection.

Clearly, $\frac{dy}{dx} > 0$ in the left neighbourhood of $\frac{1}{4}$ and $\frac{dy}{dx} < 0$ in the right neighbourhood of $\frac{1}{4}$. So, $f'(x)$ changes its sign from positive to negative in the neighbourhood of $\frac{1}{4}$.

Hence, it attains maximum at $x = \frac{1}{4}$.

TRY THESE

1. $f(x) = \int_{\cos x}^{\sin x} (1 - t + 2t^3)\,dt$ in the interval $[0, 2\pi]$, has

(a) a maximum at $\frac{\pi}{4}$ and a minimum at $\frac{3\pi}{4}$

(b) a maximum at $\frac{3\pi}{4}$ and a minimum at $\frac{7\pi}{4}$

(c) a maximum at $\frac{5\pi}{4}$ and a minimum at $\frac{7\pi}{4}$

(d) neither a maximum nor a minimum

2. If $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, $x \in [1, 3]$, then which one of the following is incorrect?

(a) $g(x)$ has a local maxima at $x = 1 + \log_e 2$ and local minima at $x = e$

(b) $f(x)$ has a local maxima at $x = 1$ and local minima at $x = 2$

(c) $f(x)$ and $g(x)$ have same points of local maxima and local minima

(d) None of the above

3. Let $f(x)$ be a function defined as $f(x) = \begin{cases} \sin(x^2 - 3x), & x \le 0 \\ 6x + 5x^2, & x > 0 \end{cases}$.

Then, at $x = 0$, $f(x)$

(a) has a local maximum

(b) has a local minimum

(c) is discontinuous

(d) None of the above

1. (b) **2.** (c) **3.** (b)

4 Using differential, the approximate value of $\sqrt{25.2}$ is

(a) 5.01 (b) 5.02 (c) 5.03 (d) 5.04

(b) ***Step I*** Define a functional relationship between the independent variable x and dependent variable y by observing the given expression.

$y = f(x) = \sqrt{x}$

Step II Choose a value of x nearest to the value for which we have to find y in such a way that either y is given for the chosen x or y can be easily computed for chosen x.

Let $x = 25$

Step III Denote the value of x at which we have to find y by $x + \Delta x$ and find Δx.

$\therefore \quad x + \Delta x = 25.2 \Rightarrow \Delta x = 25.2 - 25 = 0.2$

For $x = 25$, we have

$y = \sqrt{25} = 5$

Step IV Find $\frac{dy}{dx}$ from the relation obtained in Step I and put the value of x chosen in Step II.

Now, $y = \sqrt{x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \left(\frac{dy}{dx}\right)_{x=25} = \frac{1}{2\sqrt{25}} = \frac{1}{2(5)} = \frac{1}{10}$

Step V Find Δy, by using the relation $\Delta y = \frac{dy}{dx}\Delta x$.

$\therefore \Delta y = \frac{1}{10} \times 0.2 = 0.02$

Step VI The approximate value of y is $y + \Delta y$.

Hence, $\sqrt{25.2} = y + \Delta y$

$= 5 + 0.02 = 5.02$

TRY THESE

1. The approximate value of $(1.0002)^{3000}$ is

(a) 1.2 (b) 1.4 (c) 1.6 (d) 1.8

2. The value of $(127)^{1/3}$ to four decimal places is

(a) 5.0267 (b) 5.4267 (c) 5.5267 (d) 5.001

3. The semi-vertical angle of a cone is 45°. If the height of the cone is 20.025, then its approximate lateral surface area is

(a) $401\sqrt{2}\pi$ (b) $400\sqrt{2}\pi$

(c) $399\sqrt{2}\pi$ (d) None of these

1. (c) **2.** (a) **3.** (a)

CONCEPT MAP your Revision Tool

INTEGRATION

1 FUNDAMENTAL FORMULAE ON INTEGRATION

Following are the fundamental formulae on integration:

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int \frac{1}{x} dx = \log|x| + C$

(iii) $\int e^x dx = e^x + C$

(iv) $\int a^x dx = \frac{a^x}{\log a} + C, a > 0, a \neq 1$

(v) $\int \sin x \, dx = -\cos x + C$

(vi) $\int \cos x \, dx = \sin x + C$

(vii) $\int \sec^2 x \, dx = \tan x + C$

(viii) $\int \text{cosec}^2 x \, dx = -\cot x + C$

(ix) $\int \sec x \tan x \, dx = \sec x + C$

(x) $\int \text{cosec } x \cot x \, dx = -\text{cosec } x + C$

(xi) $\int \cot x \, dx = \log|\sin x| + C$

(xii) $\int \tan x \, dx = -\log|\cos x| + C$

(xiii) $\int \sec x \, dx = \log|\sec x + \tan x| + C$

(xiv) $\int \text{cosec } x \, dx = \log|\text{cosec } x - \cot x| + C$

(xv) $\int f(ax+b) \, dx = \frac{F(ax+b)}{a} + C$

where, $\int f(x) \, dx = F(x) + C$

2 INTEGRATION BY SUBSTITUTION

If $\phi(x)$ is a continuous differentiable function, then to evaluate integrals of the form $\int f(\phi(x))\, \phi'(x) \, dx$, we substitute $\phi(x) = t$ and $\phi'(x) \, dx = dt$

This substitution reduces the integral $\int f(\phi(x))\, \phi'(x) \, dx$ to the form $\int f(t) dt$. This integral can be evaluated by using the standard results. After evaluating this integral, we substitute back the value of t as $\phi(x)$.

3 INTEGRALS OF THE FORM

$\int \sin^m x \cos^n x \, dx, \int \sin^m x \, dx, \int \cos^n x \, dx$

If $m, n \in N$, then to evaluate this type of integrals, we use the following steps:

***Step* I** Find m and n.

***Step* II** If m is odd, i.e. power or index of $\sin x$ is odd, put $\cos x = t$ and reduce the integrand in terms of t. If n is odd, i.e. power of $\cos x$ is odd, then put $\sin x = t$ and reduce the integrand in terms of t.

If m and n both are odd, then either of the above substitutions can be used.

***Step* III** Evaluate the integral obtained in Step II and replace t by its value.

4 INTEGRALS OF THE FORM

$\int \sin^m x \cos^n x \, dx$

where, $(m+n)$ is a negative even integer. In order to evaluate this type of integrals, we use the following steps:

***Step* I** Change the integrand in terms of $\tan x$ and $\sec^2 x$ by dividing numerator and denominator by $\cos^k x$, where $k = -(m+n)$.

***Step* II** Substitute $\tan x = t$.

5 SOME STANDARD SUBSTITUTION

The following substitutions are very useful to evaluate these integrals.

Expression	Substitution
$a^2 + x^2$	$x = a \tan\theta, a \cot\theta$
$a^2 - x^2$	$x = a \sin\theta, a \cos\theta$
$x^2 - a^2$	$x = a \sec\theta, a \text{ cosec } \theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$	$x = \alpha \cos^2\theta + \beta \sin^2\theta$

Using above substitution, we can derive following results

Result 1. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

Result 2. $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

Result 3. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

Result 4. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \log|x + \sqrt{a^2+x^2}| + C$

Result 5. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

Result 6. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log|x + \sqrt{x^2-a^2}| + C$

6 INTEGRALS OF THE FORM

$\int \frac{1}{ax^2+bx+c} dx$

In order to evaluate this type of integrals, we use the following steps.

***Step* I** Make the coefficient of x^2 unity, if it is not, by multiplying and dividing by the same.

INTEGRATION BY PARTS

Let u and v be two functions of x, then

$\int uv \, dx = u\left(\int v \, dx\right) - \int \left\{\frac{du}{dx} \int v \, dx\right\} dx$

i.e. the integral of the product of two functions = (First function) × (integral of second function) − integral of {(differentiation of first function) × (integral of second function)}

Note 1. Proper choice of first and second function

Integration with the help of the above rule is called the integration by parts. In the above rule, there are two terms on RHS and in both the terms, the integral of the second function is involved. Therefore, in the product of two functions, if one of the two functions is not directly integrable (e.g. $\log x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc.), we take it as the first function and the remaining

***Step* II** Add and subtract the square of the half of the coefficient of x to express $ax^2 + bx + c$ in the form

$$a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}.$$

***Step* III** Use the suitable formula from the following formulae $\int \frac{1}{a^2+x^2} dx, \int \frac{1}{a^2-x^2} dx, \int \frac{1}{x^2-a^2} dx$

7 INTEGRALS OF THE FORM

$\int \frac{1}{\sqrt{ax^2+bx+c}} dx$

In order to evaluate this type of integrals, we use the following steps:

***Step* I** Make the coefficient of x^2 unity, if it is not.

***Step* II** Find half of the coefficient of x.

***Step* III** Add and subtract $\left(\frac{1}{2} \text{ coefficient of } x\right)^2$ inside the square root to express the quantity inside the square root in the form $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$ or $\frac{4ac-b^2}{4a^2} - \left(x + \frac{b}{2a}\right)^2$.

***Step* IV** Use the suitable formula from the following formulae

$\int \frac{1}{\sqrt{a^2+x^2}} dx, \int \frac{1}{\sqrt{x^2-a^2}} dx, \int \frac{1}{\sqrt{a^2-x^2}} dx$

8 INTEGRALS OF THE FORM

$\int \frac{px+q}{ax^2+bx+c} dx$

In order to evaluate this type of integrals, we use the following steps:

***Step* I** Write the numerator $px + q$ in the form

$$px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$$

i.e. $px + q = \lambda(2ax + b) + \mu$

***Step* II** Obtain the values of λ and μ by equating the coefficients of like powers of x and constant terms on both sides.

function is taken as the second function. If there is no other function, then unity is taken as the second function. If in the integral, both the functions are easily integrable, then the first function is chosen in such a way that the derivative of the function is a simple function and the function thus obtained under the integral sign is easily integrable than the original function.

Note 2. We can also choose the first function as the function which comes first in the word ILATE, where

I stands for the inverse trigonometric function $(\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$, etc.)

L stands for the logarithmic functions

A stands for the algebraic functions

T stands for the trigonometric functions

E stands for the exponential functions.

***Step* III** Replace $px+q$ by $\lambda(2ax+b)+\mu$ in the given integral to get

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \lambda\int \frac{2ax+b}{ax^2+bx+c}\,dx + \mu\int \frac{1}{ax^2+bx+c}\,dx$$

***Step* IV** Integrate RHS in Step III and put the values of λ and μ obtained in Step II.

Remark In order to evaluate integrals of the form $\int \frac{P(x)}{ax^2+bx+c}\,dx$, where $P(x)$ is a polynomial of degree greater than or equal to two, we divide the numerator by the denominator and express the integrand as $Q(x)+\frac{R(x)}{ax^2+bx+c}$

where, $R(x)$ is a linear function of x.

$$\therefore \quad \int \frac{P(x)}{ax^2+bx+c}\,dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}\,dx$$

9 INTEGRALS OF THE FORM

$$\int \frac{1}{a\sin^2x+b\cos^2x}\,dx, \int \frac{1}{a+b\sin^2x}\,dx,$$
$$\int \frac{1}{a+b\cos^2x}\,dx, \int \frac{1}{(a\sin x+b\cos x)^2}\,dx,$$
$$\int \frac{1}{a+b\sin^2x+c\cos^2x}\,dx$$

To evaluate this type of integrals, we use the following steps:

***Step* I** Divide numerator and denominator by $\cos^2x$.

***Step* II** Replace $\sec^2x$, if any, in denominator by $1+\tan^2x$.

***Step* III** Put $\tan x = t$ so that $\sec^2x\,dx = dt$.

***Step* IV** After performing the above steps, the integral will reduce to the form $\int \frac{1}{at^2+bt+c}\,dt$ which can be integrated.

10 INTEGRALS OF THE FORM

$$\int \frac{1}{a\sin x+b\cos x}\,dx, \int \frac{1}{a+b\sin x}\,dx,$$
$$\int \frac{1}{a+b\cos x}\,dx, \int \frac{1}{a\sin x+b\cos x+c}\,dx$$

To evaluate this type of integrals, we use the following steps:

***Step* I** Put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$,

$$\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}.$$

***Step* II** Replace $1+\tan^2\frac{x}{2}$ in the numerator by $\sec^2\frac{x}{2}$.

***Step* III** Put $\tan\frac{x}{2}=t$ so that $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$.

After performing these three steps, the integral reduces to the form $\int \frac{1}{at^2+bt+c}\,dt$ which can be integrated.

To evaluate integrals of the form

$$\int \frac{1}{a\sin x+b\cos x}\,dx,$$

we may also substitute $a = r\cos\theta, b = r\sin\theta$ and so $r=\sqrt{a^2+b^2}, \theta = \tan^{-1}\left(\frac{b}{a}\right)$.

$$\therefore \quad I = \int \frac{1}{a\sin x+b\cos x}\,dx = \frac{1}{r}\int \frac{1}{\sin(x+\theta)}\,dx$$

$$\Rightarrow \quad I = \frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx = \frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| + C$$

$$\Rightarrow I = \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right| + C$$

11 INTEGRALS OF THE FORM $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}\,dx$

To evaluate this type of integrals, we express the numerator as follows:

Numerator $=\lambda$ (Differentiation of denominator) $+\mu$(denominator) i.e. $(a\sin x+b\cos x)$

$$= \lambda\frac{d}{dx}(c\sin x+d\cos x)+\mu(c\sin x+d\cos x)$$

where, λ and μ are constants to be determined by comparing the coefficients of $\sin x$ and $\cos x$ on both sides.

$$\therefore \quad \int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}\,dx = \int \frac{\lambda(c\cos x-d\sin x)+\mu(c\sin x+d\cos x)}{c\sin x+d\cos x}\,dx$$

$$= \int \mu\,dx + \lambda\int \frac{c\cos x-d\sin x}{c\sin x+d\cos x}\,dx$$

$$= \mu x + \lambda\log|c\sin x+d\cos x| + K$$

12 INTEGRALS OF THE FORM $\int e^x\{f(x)+f'(x)\}\,dx$

Theorem 1. $\int e^x\{f(x)+f'(x)\}\,dx = e^xf(x)+C$

Note The above theorem is also true, if we have e^{kx} in place of e^x.

i.e. $\int e^{kx}\{k\,f(x)+f'(x)\}\,dx = e^{kx}\,f(x)+C$

Theorem 2.

(i) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$

(ii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$

13 SOME IMPORTANT INTEGRALS

The following are some useful integrals.

(i) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+C$

(ii) $\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\log\left|x+\sqrt{a^2+x^2}\right|+C$

(iii) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right|+C$

Note To evaluate integrals of the form $\int\sqrt{ax^2+bx+c}\,dx$, we express ax^2+bx+c as the sum or difference of two squares and apply one of the formulae given above as the case may be.

14 INTEGRALS OF THE FORM $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

To evaluate this type of integrals, we express the linear factor $px+q$ as $px+q = \lambda\frac{d}{dx}(ax^2+bx+c)+\mu$

where, λ and μ can be determined by equating the coefficients of like powers of x and constant term on both sides.

$$\therefore \int (px+q)\sqrt{ax^2+bx+c}\,dx = \lambda\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \mu\int \sqrt{ax^2+b+c}\,dx$$

The first integral on RHS is evaluated by putting $ax^2+bx+c=t$ and the second integral is evaluated by the method discussed above.

15 INTEGRALS OF THE FORM

$$\int \frac{x^2+1}{x^4+\lambda x^2+1}\,dx, \int \frac{x^2-1}{x^4+\lambda x^2+1}\,dx,$$
$$\int \frac{1}{x^4+\lambda x^2+1}\,dx$$

To evaluate this type of integrals, we divide the numerator and denominator by x^2 and put $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$, whichever on differentiation gives the numerator of the resulting integrand.

Er. Love Agrawal

JEE ADVANCED DRILL

A collection of 30 innovative problems on JEE Advanced pattern from complete syllabus

Single Option Correct Type

1. If a circle of constant radius r passes through the origin O and cuts the axes at P and Q. Then, the locus of the perpendicular from O to PQ is

(a) $(x^2 - y^2)^2(x^{-2} + y^{-2}) = 4r^2$

(b) $(x^2 + y^2)^2(x^{-2} + y^{-2}) = 4r^2$

(c) $(x^2 + y^2)^2(x^{-2} - y^{-2}) = 4r^2$

(d) $(x^2 - y^2)^2(x^{-2} - y^{-2}) = 4r^2$

2. The equation of the curve for which the intercept cut-off by any tangent on Y-axis is proportional to the square of the ordinate of the point of tangeney, is

(a) $\frac{C_1}{x} + \frac{C_2}{y} = 1$ *(b)* $\frac{C_1}{x} - \frac{C_2}{y} = 1$

(c) $\frac{1}{x} + \frac{1}{y} = C$ *(d)* $\frac{1}{x} - \frac{1}{y} = C$

3. The value of $\lim_{n\to\infty} \frac{1}{n^2}\{[x] + [2x] + [3x] + \ldots [nx]\}$, where $[\cdot]$ denotes greatest integer function, is

(a) $\frac{x}{4}$ *(b)* $\frac{x}{3}$ *(c)* $\frac{x}{2}$ *(d)* $\frac{x}{5}$

4. The points on the curve $ax^2 + 2bxy + ay^2 = c$, $0 < a < b < c$, whose distance from the origin is minimum, is

(a) $\pm\left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$

(b) $\pm\left(\sqrt{\frac{a}{2(b+c)}}, \sqrt{\frac{a}{2(b+c)}}\right)$

(c) $\pm\left(\sqrt{\frac{b}{2(a+c)}}, \sqrt{\frac{b}{2(a+c)}}\right)$

(d) $\pm\left(\sqrt{\frac{a}{(a+b+c)}}, \sqrt{\frac{c}{(a+b+c)}}\right)$

5. The sum $\sum_{r=1}^{n} \text{arc tan}\left(\frac{2r}{2 + r^2 + r^4}\right)$ is equal to

(a) $\tan^{-1}(n^2 + n + 1) + \frac{\pi}{4}$ *(b)* $\tan^{-1}(n^2 + n + 1) - \frac{\pi}{4}$

(c) $\tan^{-1}(n^2 + 2n + 1) - \frac{\pi}{4}$ *(d)* $\tan^{-1}(n^2 - n + 1) - \frac{\pi}{4}$

More Than One Option Correct Type

6. If $|z_1| = |z_2| = 1$ and $\arg(z_1) + \arg(z_2) = 0$, then

(a) $z_1 + z_2 = 0$ *(b)* $z_1 z_2 = 1$ *(c)* $\bar{z}_1 = z_1$ *(d)* $z_1 = \bar{z}_2$

7. Let $(1 + x^2)^2(1 + x)^n = \sum_{r=0}^{n+4} a_r x^r$. If a_1, a_2, a_3 are in AP, then the possible values of n are

(a) 2 *(b)* 5 *(c)* 4 *(d)* 3

8. If $a^x = b$, $b^y = c$, $c^z = a$, $x = \log_b a^{k_1}$, $y = \log_c b^{k_2}$ and $z = \log_a c^{k_3}$, then

(a) $xyz = 1$ *(b)* $xyz = 2$ *(c)* $k_1k_2k_3 = 2$ *(d)* $k_1k_2k_3 = 1$

9. If the vectors $\mathbf{b} = (\tan\theta, -1, 2\sqrt{\sin\theta/2}$ and $\mathbf{c} = \left(\tan\theta, \tan\theta, -\frac{3}{\sqrt{\sin\theta/2}}\right)$ are orthogonal and a vector $\mathbf{a} = (1, 3\sin 2\theta)$ makes an obtuse angle with the Z-axis, then the value of θ is

(a) $(4n + 2)\pi - \tan^{-1} 2$ *(b)* $(4n + 2)\pi + \tan^{-1} 2$

(c) $(4n + 1)\pi - \tan^{-1} 2$ *(d)* $(4n + 1)\pi + \tan^{-1} 2$

10. If an ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal to that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then

(a) equation of ellipse is $x^2 - 2y^2 = 2$

(b) the foci of ellipse is $(\pm 1, 0)$

(c) equation of ellipse is $x^2 + 2y^2 = 2$

(d) the foci of ellipse is $(\pm\sqrt{3}, 0)$

11. If a function is defined as $f(x)=\begin{cases} e^x, & x\le 0 \\ |x-2|, & x>0 \end{cases}$, then

(a) $f(x)$ is continuous at $x=0$ and 2 both
(b) $f(x)$ is continuous at $x=2$
(c) $f(x)$ is not differentiable at $x=2$
(d) $f(x)$ is differentiable at $x=0$

12. If a function $f:R\longrightarrow[-1,1]$ is defined as $f(x)=\dfrac{x^2-1}{x^2+1}$, then which of the following statement is true?

(a) f is increasing for $x>0$ and decreasing for $x<0$
(b) the area included by the curve $y=f(x)$ and the line $y=1$ is 2π sq units
(c) minimum value is not attained even though f is bounded
(d) f is many-one and onto

13. Three six faced fair dice are thrown together. The probability that the sum of the numbers appearing on the dice is $n(3\le n\le 8)$ is

(a) $\dfrac{n^2}{432}$ (b) $\dfrac{(n-1)(n-2)}{432}$
(c) $\dfrac{n^2-3n+2}{432}$ (d) ${}^{n-1}C_2\times\dfrac{-1}{216}$

Single Integer Type

14. The value of p for which the numbers $\log_5 2, \log_5(2^p-5)$ and $\log_5\left(2^p-\dfrac{7}{2}\right)$ are in AP, is … .

15. If $f(x)=\begin{vmatrix} \cos x & x^2 & 1 \\ e^{x^2} & \sec x & 2 \\ 2x\cos^2 x/2 & x^3+\sin x & x+\tan x \end{vmatrix}$, then the value of $\int_{-\pi/2}^{\pi/2}(f(x)+f''(x))(x^2+1)dx$ is … .

16. If $a_1\ne p_1, a_2\ne p_2, a_3\ne p_3$ and the system of equations $p_1x+a_2y+a_3z=0$, $a_1x+p_2y+a_3z=0$ and $a_1x+a_2y+p_3z=0$ has a non-zero solution, then the value of $\dfrac{p_1}{p_1-a_1}+\dfrac{p_2}{p_2-a_2}+\dfrac{p_3}{p_3-a_3}$ is … .

17. If $f(x+y)=f(x)+f(y)-2xy-1$ for all x and y and $f'(0)=-\sin\alpha$, then the value of $f\{f'(0)\}$ is ……

18. If $\{x\}$ and $[x]$ denote the fractional and integral parts of a positive real number x respectively, then the number of solutions of the equation $5\{x\}=x+[x]$ is …….

19. Points S and T are taken on the sides QR of ΔPQR such that $QS=ST=TR$. If $\angle QPS=\theta_1, \angle SPT=\theta_2$, $\angle TPR=\theta_3$, then the value of $\dfrac{\sin(\theta_1+\theta_2)\sin(\theta_2+\theta_3)}{\sin\theta_1\cdot\sin\theta_3}$ is equal to … .

20. If $\sum_{r=1}^{n}t_r=\dfrac{n(n+1)(n+2)(n+3)}{8}$, then the value of $21\sum_{r=1}^{5}\dfrac{1}{2t_r}$ is … .

21. If the points $\left(\dfrac{a^3}{a-1},\dfrac{a^2-3}{a-1}\right)$, $\left(\dfrac{b^3}{b-1},\dfrac{b^2-3}{b-1}\right)$ and $\left(\dfrac{c^3}{c-1},\dfrac{c^2-3}{c-1}\right)$ are collinear for three distinct values a, b, c and different from 1, then the value of $abc-(bc+ca+ab)+3(a+b+c)$ is equal to …

Linked Comprehension Type

Directions (Q. Nos. 22-23) A curve $y=f(x)$ satisfies the differential equation $\dfrac{dy}{dx}+\left(\dfrac{2x}{1+x^2}\right)y=\dfrac{4x^2}{1+x^2}$ and passes through the origin.

22. The function $y=f(x)$

(a) has no inflection point
(b) is strictly decreasing, $\forall\ x\in R$
(c) is strictly increasing, $\forall\ x\in R$
(d) is such that it has a maxima but no minima

23. The area enclosed by $y=f(x)$, X-axis and the ordinate at $x=\dfrac{2}{3}$, is

(a) $\log 2$ (b) $\dfrac{4}{3}\log 2$ (c) $\dfrac{2}{3}\log 2$ (d) $\dfrac{1}{3}\log 2$

Directions (Q. Nos. 24-25) The normal at a point $P(am^2,-2am)$ intersects the parabola $y^2=4ax$ at point Q. If the tangents at P and Q meet at R, then

24. The coordinates of R is

(a) $\left(2a-am^2,\dfrac{2a}{m}\right)$ (b) $\left(-2a-am^2,\dfrac{2a}{m}\right)$
(c) $\left(-2a-am^2,\dfrac{a}{m}\right)$ (d) $\left(2a+am^2,\dfrac{2a}{m}\right)$

25. The area of ΔPQR is

(a) $\dfrac{4a^2(1+m^2)^3}{m^3}$ (b) $\dfrac{4a^2(1+m^2)}{m^3}$
(c) $\dfrac{4a(1+m^2)^3}{m^3}$ (d) $\dfrac{a^2(1+m^2)^3}{m^3}$

Directions (Q. Nos. 26-27) A bag contains p red and q blue balls. Two players P and Q alternatively draw a ball from the bag, replacing the ball each time after the draw till one of them draws a red ball and wins the game. If P begins the game and the probability of A winning the game is three times that of B.

26. The probability of Q wins the game is

(a) $\dfrac{p}{p+2q}$ (b) $\dfrac{q}{2p+q}$ (c) $\dfrac{q}{p+2q}$ (d) $\dfrac{p}{2p+q}$

27. The ratio of p and q is

(a) 1 : 2 (b) 2 : 1 (c) 1 : 3 (d) 3 : 1

Match the Columns

28. Match the following columns.

Column I	Column II
A. If $(3, \lambda, \mu)$ is a point one the lines $2x + y + z - 3 = 0 = x - 2y + z - 1$, then $\lambda + \mu$ is equal to	p. $(-1, -1, -1)$
B. Image of the point $(3, 5, 7)$ in the plane $2x + y + z = -18$ is	q. 9
C. The distance of the point $(2, 3, 4)$ from the plane $3x + 2y + 2z + 5 = 0$, measured parallel to the line $\frac{x+3}{3} = \frac{y-2}{3} = \frac{z}{2}$, is	r. -3
D. The intersection point of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$, is	s. $(-21, -7, -5)$

29. Match the following columns.

Column I	Column II
A. If M is a 3×3 matrix, where $M^T M = I$ and $\det(M) = 1$, then $\det(M - I)$ is equal to	p. 1
B. If $\begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda^2+1 & -2\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix} = a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e$ is an identity in λ, where a, b, c, d and e are constants, then the value of $e + 1$ is	q. $\frac{2}{5}$
C. If α, β and γ are three real numbers, then $\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{vmatrix}$ is	r. 0
D. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$ and its inverse is denoted by $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the value of a_{23} is	s. 0

30. Match the following columns.

Column I	Column II
A. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, then	p. $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are coplanar
B. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are unit vectors such that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 1$, then	q. $\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}, \mathbf{c} + \mathbf{a}$ are also coplanar
C. Angle between vectors $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit vectors satisfying $\mathbf{a} + \mathbf{b} + \sqrt{3}\mathbf{c} = 0$, is	r. Obtuse angle
D. If $\lvert\mathbf{a} \times \mathbf{b}\rvert = \lvert\mathbf{a} + 2\mathbf{b}\rvert$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is	s. $\frac{\pi}{3}$

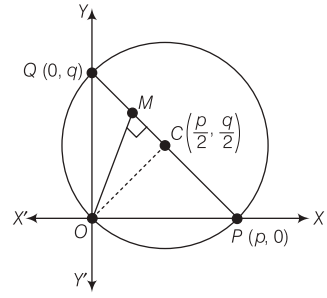
Answers with Explanation

1. ***(b)*** Let the coordinates of P and Q be $(p, 0)$ and $(0, q)$.

Equation of PQ is

$$\frac{x}{p} + \frac{y}{q} = 1 \qquad \ldots(i)$$

Centre of the circle lies on the line PQ, since PQ is a diameter of the circle.



$\therefore$ Radius, $r = PC = OC = QC$

$$= \sqrt{\left(0 - \frac{p}{2}\right)^2 + \left(0 - \frac{q}{2}\right)^2} = \sqrt{\frac{p^2 + q^2}{4}}$$

$\Rightarrow \quad p^2 + q^2 = 4r^2 \qquad \ldots(ii)$

Equation of OM which is perpendicular to PQ is

$$px = qy = \lambda$$

It passes through $(0, 0)$.

$\therefore \quad \lambda = 0$

$\therefore$ Equation of OM is $px - qy = 0 \qquad \ldots(iii)$

From Eqs. (i) and (ii), we get

$$p = \frac{x^2 + y^2}{x} \text{ and } q = \frac{x^2 + y^2}{y}$$

On substituting the values of p and q in Eq. (ii), we get

$$(x^2 + y^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4r^2$$

$\Rightarrow \quad (x^2 + y^2)^2 (x^{-2} + y^{-2}) = 4r^2$

2. ***(a)*** The equation of tangent at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

When $X = 0$, then $Y = y - x\frac{dy}{dx}$ [y-intercept]

It is given that $Y \propto y^2$

$\therefore \quad y - x\frac{dy}{dx} = ky^2$ [where k is the proportional constant]

$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = -\frac{ky^2}{x}$

$\Rightarrow \quad -y^{-2}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} = \frac{k}{x}$

Let $\quad \frac{1}{y} = t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \quad \frac{dt}{dx} + \frac{1}{x} \cdot t = \frac{k}{x}$

Now, $\quad \text{IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Hence, required solution is

$$tx = \int \frac{kx}{x}dx = kx + C \Rightarrow \frac{x}{y} = kx + C$$

$$\Rightarrow \quad x = kxy + Cy \Rightarrow x - Cy = kxy$$

$$\Rightarrow \quad \frac{C_1}{x} + \frac{C_2}{y} = 1$$

3. ***(c)*** Since, $x - 1 < [x] \le x$

$$\therefore \quad 2x - 1 < [2x] \le 2x$$
$$3x - 1 < [3x] \le 3x$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$nx - 1 < [nx] \le nx$$

On adding above terms, we get

$$x(1 + 2 + 3 + \ldots + n) - n < [x] + [2x] + [3x] + \ldots + [nx] \le x(1 + 2 + 3 + \ldots + n)$$

$$\Rightarrow \quad x\frac{n(n+1)}{2} - n < [x] + [2x] + [3x] + \ldots + [nx] \le x\frac{n(n+1)}{2}$$

$$\Rightarrow \frac{x\frac{n(n+1)}{2} - n}{n^2} < \frac{[x] + [2x] + [3x] + \ldots + [nx]}{n^2} \le \frac{x\frac{n(n+1)}{2}}{n^2}$$

$$\Rightarrow \lim_{n\to\infty} x\left(\frac{n+1}{2} - 1\right)\frac{1}{n} < \lim_{n\to\infty} \frac{[x] + [2x] + [3x] + \ldots + [nx]}{n^2}$$
$$\le \lim_{n\to\infty} x\frac{n(n+1)}{2}$$

$$\Rightarrow \quad \frac{x}{2} < \lim_{n\to\infty} \frac{[x] + [2x] + \ldots + [nx]}{n^2} \le \frac{x}{2}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{[x] + [2x] + \ldots + [nx]}{n^2} = \frac{x}{2}$$

4. ***(a)*** Let $P(x, y)$ be a point on the curve

$$ax^2 + 2bxy + ay^2 = c$$

whose distance from the origin is r.

$$\therefore \quad x = r\cos\theta \text{ and } y = r\sin\theta$$

As $r\cos\theta$ and $r\sin\theta$ lie on $ax^2 + 2bxy + ay^2 = c$.

$$\therefore \quad ar^2\cos^2\theta + 2br^2\sin\theta\cos\theta + ar^2\sin^2\theta = c$$

$$\Rightarrow \quad (a + b\sin 2\theta)r^2 = c$$

$$\Rightarrow \quad r^2 = \frac{c}{a + b\sin 2\theta} \qquad \ldots(i)$$

From Eq. (i), r is minimum when $(a + b\sin 2\theta)$ is maximum.

$\Rightarrow \sin 2\theta$ is maximum.

$$\Rightarrow \quad 2\theta = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

For θ, maximum value of $(a + b\sin 2\theta) = a + b$

$$\therefore \quad r_{\min} = \sqrt{\frac{c}{a+b}}$$

When $\theta = \frac{\pi}{4}$, $P\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right) = \left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$

When $\theta = \frac{5\pi}{4}$, $P\left(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right) = \left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$

5. ***(b)*** $\sum_{r=1}^{n} \arctan\left(\frac{2r}{2 + r^2 + r^4}\right)$

$$= \sum_{r=1}^{n} \tan^{-1}\left\{\frac{(r^2 + r + 1) - (r^2 - r + 1)}{1 + (r^2 + r + 1)(r^2 - r + 1)}\right\}$$

$$= \sum_{r=1}^{n} \tan^{-1}(r^2 + r + 1) - \sum_{r=1}^{n} \tan^{-1}(r^2 - r + 1)$$

$$= [\tan^{-1} 3 + \tan^{-1} 7 + \tan^{-1} 13 + \ldots + \tan^{-1}(n^2 - n + 1) + \tan^{-1}(n^2 + n + 1)] - [\tan^{-1} 1 + \tan^{-1} 3 + \tan^{-1} 7 + \ldots + \tan^{-1}(n^2 - n + 1)]$$

$$= \tan^{-1}(n^2 + n + 1) - \tan^{-1} 1$$

$$= \tan^{-1}(n^2 + n + 1) - \frac{\pi}{4}$$

6. ***(b, d)*** Let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\Rightarrow \quad |z_2| = r_2$$

Since, $\arg(z_1) + \arg(z_2) = 0$

$$\Rightarrow \quad \arg(z_1) = -\arg(z_2) = -\theta_2$$

$$\therefore \quad z_1 = r_2[\cos(-\theta_2) + i\sin(-\theta_2)]$$
$$= r_2[\cos\theta_2 - i\sin\theta_2] = \bar{z}_2$$

$$\Rightarrow \quad z_1 = \frac{1}{z_2} \Rightarrow z_1 z_2 = 1$$

7. ***(a, c, d)*** We have,

$$(1 + 2x^2 + x^4)(1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots)$$
$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

On comparing the coefficients of x, x^2, x^3, we get

$a_1 = {}^nC_1$, $a_2 = {}^nC_2 + 2$ and $a_3 = {}^nC_3 + 2\,{}^nC_1$

Since, a_1, a_2 and a_3 are in AP.

$$\therefore \quad 2a_2 = a_1 + a_3 \Rightarrow 2({}^nC_2 + 2) = {}^nC_1 + {}^nC_3 + 2\,{}^nC_1$$

$$\Rightarrow \quad 2\frac{n(n-1)}{2} + 4 = 3n + \frac{n(n-1)(n-2)}{6}$$

$$\Rightarrow \quad n^2 - n + 4 = 3n + \frac{n(n^2 - 2n - n + 2)}{6}$$

$$\Rightarrow \quad 6(n^2 - n + 4) = 18n + n^3 - 3n^2 + 2n$$

$$\Rightarrow \quad n^3 - 9n^2 + 26n - 24 = 0$$

$$\Rightarrow \quad (n-2)(n-3)(n-4) = 0$$

$$\Rightarrow \quad n = 2, 3, 4$$

8. ***(a, d)*** Since, $c^z = a$

$$\Rightarrow \quad (b^y)^z = a \Rightarrow b^{yz} = a$$

$$\Rightarrow \quad (a^x)^{yz} = a \Rightarrow a^{xyz} = a$$

$$\Rightarrow \quad xyz = 1$$

Again, $x = \log_a b$, $y = \log_b c$ and $z = \log_c a$

$$\Rightarrow \quad \log_b a^{k_1} = \log_a b, \log_c b^{k_2} = \log_b c,$$

and $\quad \log_a c^{k_3} = \log_c a$

$$\Rightarrow \quad k_1 = \frac{\log_a b}{\log_b a}, k_2 = \frac{\log_b c}{\log_c b}, k_3 = \frac{\log_c a}{\log_a c}$$

$$\Rightarrow \quad k_1 = (\log_a b)^2, k_2 = (\log_b c)^2, k_3 = (\log_c a)^2$$

$$\Rightarrow \quad k_1 = x^2, k_2 = y^2 \text{ and } k_3 = z^2$$

Hence, $\quad k_1 k_2 k_3 = (xyz)^2 = 1$

9. ***(a, c)*** Given, $\mathbf{a} = (1, 3, \sin 2\theta)$ makes an obtuse angle with the Z-axis, therefore its Z-component is negative.

$$\Rightarrow \quad \sin 2\theta < 0 \Rightarrow -1 \le \sin 2\theta < 1 \qquad \ldots(i)$$

Since, $\mathbf{b}$ and $\mathbf{c}$ are orthogonal.

$$\Rightarrow \quad \mathbf{b} \cdot \mathbf{c} = 0$$

$$\Rightarrow \quad \tan^2\theta - \tan\theta - 6 = 0 \Rightarrow (\tan\theta - 3)(\tan\theta + 2) = 0$$

$\Rightarrow \quad \tan\theta = -2, 3$

Now, $\tan\theta = 3$

$\Rightarrow \quad \sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{6}{1+9} = \dfrac{3}{5} > 0$

which is not possible, since $\sin 2\theta < 0$.

Now, if $\tan\theta = -2$

$\Rightarrow \quad \sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{-4}{1+4} = \dfrac{-4}{5} < 0$

which is true.

$\therefore \quad \tan 2\theta > 0$

$\therefore$ 2θ is in third quadrant and $\sqrt{\sin\theta/2}$ is meaningful.

If $0 < \sin\theta/2 < 1$, then

$\theta = (4n+1)\pi - \tan^{-1} 2$

and $\theta = (4n+2)\pi - \tan^{-1} 2$

10. ***(b, c)*** Since, given hyperbola is rectangular, so its eccentricity is $\sqrt{2}$ and eccentricity e of the ellipse is $\dfrac{1}{\sqrt{2}}$.

Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,

where $b^2 = a^2(1-e^2) = \dfrac{a^2}{2} \Rightarrow a^2 = 2b^2$

So, equation of the ellipse is

$x^2 + 2y^2 = a^2$

and let (x_1, y_1) be a point of intersection of the ellipse and the hyperbola.

Then, $2x_1^2 - 2y_1^2 = 1$ and $x_1^2 + 2y_1^2 = a^2$ ...(i)

Equations of the tangents at (x_1, y_1) to the two conics are

$2xx_1 - 2yy_1 = 1$ and $xx_1 + 2yy_1 = a^2$

Since, the two conics intersects orthogonally.

$\therefore \quad \left(\dfrac{x_1}{y_1}\right)\left(-\dfrac{x_1}{2y_1}\right) = -1 \Rightarrow x_1^2 = 2y_1^2$

From Eq. (i), we get $x_1^2 = 1, a^2 = 2$

Hence, the equation of ellipse is $x^2 + 2y^2 = 2$ and its focus is $(\pm ae, 0) = (\pm 1, 0)$.

11. ***(b, c)*** We have, $f(x) = \begin{cases} e^x, & x \le 0 \\ 2-x, & 0 < x < 2 \\ x-2, & x \ge 2 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} e^{-h} = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = 2$$

$\Rightarrow f(x)$ is not continuous at $x = 0$.

and $$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = 0$$

$$f(2) = 0$$

$\Rightarrow f(x)$ is continuous at $x = 2$.

Now, $f'(0^-) = 1$ and $f'(0^+) = -1$

$\Rightarrow \quad f'(2^-) = -1$ and $f'(2^+) = 1$

$\Rightarrow$ $f(x)$ is not differentiable at $x = 0, 2$.

12. ***(a, b)*** We have, $f(x) = \dfrac{x^2-1}{x^2+1}$

$$= 1 - \frac{2}{x^2+1}$$

$\Rightarrow \quad f'(x) = \dfrac{4x}{(x^2+1)^2}$

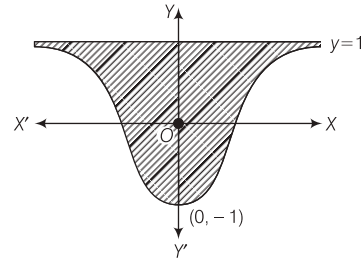
If $x > 0$, $f'(x) > 0 \Rightarrow f$ is increasing and if $x < 0$, $f'(x) < 0$ $\Rightarrow f$ is decreasing.

Range $(f) = [-1, 1]$

So, it is into function.

Minimum value occurs at $x = 0$

$\therefore \quad f(x) = f(0) = -1$



Now, $A = 2\int_0^\infty \left(1 - \dfrac{x^2-1}{x^2+1}\right) dx$

$$= 4\int_0^\infty \frac{dx}{x^2+1} dx = 4\,[\tan^{-1} x]_0^\infty$$

$$= 4 \cdot \frac{\pi}{2} = 2\pi$$

13. ***(b, c)*** Total number of cases

$= 6 \times 6 \times 6 = 216$

The number of favourable ways

= Coefficient of x^n in $(x + x^2 + \ldots + x^6)^3$

= Coefficient of x^{n-3} in $(1-x^6)^3(1-x)^{-3}$

= Coefficient of x^{n-3} in $(1-x)^{-3}$ $\quad [\because\ 0 \le n-3 \le 5]$

$= {}^{n-3+2}C_2 = {}^{n-1}C_2 = \dfrac{(n-1)(n-2)}{2}$

$\therefore$ Required probability $= \dfrac{(n-1)(n-2)}{432}$

14. ***(3)*** Given, $\log_5 2, \log_5(2^p - 5)$ and $\log_5\left(2^p - \dfrac{7}{2}\right)$ are in AP.

$$\therefore \quad \log_5(2^p - 5) = \frac{\log_5 2 + \log_5\left(2^p - \frac{7}{2}\right)}{2}$$

$$\Rightarrow \quad (2^p - 5)^2 = 2\left(2^p - \frac{7}{2}\right)$$

$\Rightarrow \quad (2^p)^2 - 10(2^p) + 25 = 2\cdot(2^p) - 7$

$\Rightarrow \quad (2^p)^2 - 12(2^p) + 32 = 0$

$\Rightarrow \quad (2^p - 8)(2^p - 4) = 0$

$\Rightarrow \quad p = 2, 3$

Since, $\log_5(2^p - 5)$ exists only when $(2^p - 5) > 0$.

$\therefore$ $p = 3$ is the only solution.

15. *(0)* We have, $f(x) = \begin{vmatrix} \cos x & x^2 & 1 \\ e^{x^2} & \sec x & 2 \\ 2x\cos^2 x/2 & x^3 + \sin x & x + \tan x \end{vmatrix}$

$\Rightarrow \quad f(-x) = -f(x)$

$\Rightarrow$ $f(x)$ is odd

$\Rightarrow$ $f'(x)$ is even

$\Rightarrow$ $f''(x)$ is odd

$\Rightarrow$ $f(x) + f''(x)$ is odd function

Let $h(x) = (x^2 + 1)\{f(x) + f''(x)\}$

$\Rightarrow \quad h(-x) = -h(x)$

$\Rightarrow$ $h(x)$ is odd

$\therefore \quad \int_{-\pi/2}^{\pi/2} h(x)dx = 0$

16. *(2)* Since, system of equations has non-zero solution and $D_1 = D_2 = D_3 = 0$. So, D must be equal to zero.

$\therefore \quad \begin{vmatrix} p_1 & a_2 & a_3 \\ a_1 & p_2 & a_3 \\ a_1 & a_2 & p_3 \end{vmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} p_1 - a_1 & a_2 - p_2 & 0 \\ 0 & p_2 - a_2 & a_3 - p_3 \\ a_1 & a_2 & p_3 \end{vmatrix} = 0$

$[\because R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$

$\Rightarrow (p_1 - a_1)(p_2 - a_2)(p_3 - a_3)$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a_1}{p_1 - a_1} & \frac{a_2}{p_2 - a_2} & \frac{p_3}{p_3 - a_3} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a_1}{p_1 - a_1} + \frac{a_2}{p_2 - a_2} + \frac{p_3}{p_3 - a_3} & \frac{a_2}{p_2 - a_2} & \frac{p_3}{p_3 - a_3} \end{vmatrix} = 0$$

$[\because C_1 \to C_1 + C_2 + C_3]$

$$\Rightarrow \quad \frac{a_1}{p_1 - a_1} + \frac{a_2}{p_2 - a_2} + \frac{p_3}{p_3 - a_3} = 0$$

$$\Rightarrow \quad \left(\frac{a_1}{p_1 - a_1} + 1\right) + \left(\frac{a_2}{p_2 - a_2} + 1\right) + \left(\frac{p_3}{p_3 - a_3}\right) = 2$$

$$\Rightarrow \quad \frac{p_1}{p_1 - a_1} + \frac{p_2}{p_2 - a_2} + \frac{p_3}{p_3 - a_3} = 2$$

17. *(1)* $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\{f(x) + f(h) - 2xh - 1\} - f(x)}{h}$$

$$= \lim_{h \to 0} \left\{-2x + \frac{f(h) - 1}{h}\right\}$$

$$= \lim_{h \to 0} \left\{-2x + \frac{f(h) - f(0)}{h}\right\} \qquad [\because f(0) = 1]$$

$\Rightarrow \quad f'(x) = -2x + f'(0)$

$\Rightarrow \quad f(x) = -x^2 - \sin\alpha \cdot x + C$

$\Rightarrow \quad f(x) = -x^2 - \sin\alpha \cdot x + 1$

[when $x = 0$, then $C = 1$]

$\therefore \quad f\{f'(0)\} = f(-\sin\alpha) = -\sin^2\alpha + \sin^2\alpha + 1 = 1$

18. *(2)* If x is positive real number, then

$$x = [x] + \{x\} \qquad \ldots(i)$$

where $0 \le \{x\} < 1$

Now, $\quad 5\{x - [x]\} = x + [x]$

$\Rightarrow \quad 4x = 6[x]$

$\Rightarrow \quad x = \frac{3}{2}[x]$

From Eq. (i),

$$\frac{3}{2}[x] = [x] + \{x\}$$

$\Rightarrow \quad \{x\} = \frac{1}{2}[x]$

Since, $0 \le \{x\} < 1$

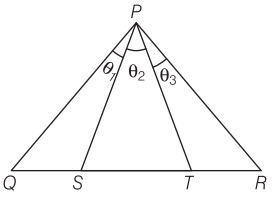
$\Rightarrow \quad 0 \le \frac{1}{2}[x] < 1$

$\Rightarrow \quad 0 \le [x] < 2$

$\Rightarrow \quad [x] = 0, 1$

$\Rightarrow \quad x = 0, \frac{1}{2}$

19. *(4)* In ΔPQS,



$$\frac{\sin\theta_1}{QS} = \frac{\sin Q}{PS} \qquad \ldots(i)$$

In ΔPQT, $\quad \frac{\sin(\theta_1 + \theta_2)}{QT} = \frac{\sin Q}{PT} \qquad \ldots(ii)$

In ΔPTR, $\quad \frac{\sin\theta_3}{TR} = \frac{\sin R}{PT} \qquad \ldots(iii)$

In ΔPSR, $\quad \frac{\sin(\theta_2 + \theta_3)}{SR} = \frac{\sin R}{PS} \qquad \ldots(iv)$

From Eqs. (i), (ii), (iii) and (iv),

$$\frac{\sin(\theta_1 + \theta_2)\sin(\theta_2 + \theta_3)}{\sin\theta_1 \cdot \sin\theta_3} = \frac{\left(\frac{QT}{PT}\sin Q\right)\left(\frac{SR}{PS}\sin R\right)}{\left(\frac{QS}{PS}\sin Q\right)\left(\frac{TR}{PT}\sin R\right)}$$

$$= 2 \times 2 = 4$$

20. *(5)* $t_n = \sum_{r=1}^{n} t_r - \sum_{r=1}^{n-1} t_r = \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)(n)(n+1)(n+2)}{8}$

$$= \frac{n(n+1)(n+2)}{8}(4) = \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \quad \frac{1}{t_n} = \frac{2}{n(n+1)(n+2)}$$

Let $v_n = \frac{2}{(n+1)(n+2)} \Rightarrow v_{n-1} = \frac{2}{n(n+1)}$

$$\Rightarrow \quad v_n - v_{n-1} = \frac{2}{(n+1)(n+2)} - \frac{2}{n(n+1)}$$

$$= -\frac{4}{n(n+1)(n+2)} = -2\left(\frac{1}{t_n}\right)$$

$$\Rightarrow \quad \frac{1}{t_n} = -\frac{1}{2}(v_n - v_{n-1})$$

$$\Rightarrow \quad \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \ldots + \frac{1}{t_n}$$

$$= -\frac{1}{2}(v_n - v_0)$$

$$= -\frac{1}{2}\left[\frac{2}{(n+1)(n+2)} - 1\right]$$

$$= \frac{n^2 + 3n}{2(n+1)(n+2)}$$

$$\therefore \quad 21\sum_{r=1}^{5}\frac{1}{2t_r} = 21\left(\frac{25+15}{2\times 2(5+1)(5+2)}\right) = 5$$

21. ***(0)*** Let the three given points lie on the line $lx + my + n = 0$, where l, m and n are real numbers.

Then, $\quad l\left(\frac{t^3}{t-1}\right) + m\left(\frac{t^2-3}{t-1}\right) + n = 0$

$$\Rightarrow \quad lt^3 + mt^2 + nt - (3m + n) = 0$$

For $t = a, b, c$

$\Rightarrow a, b, c$ are roots of equation

$$lt^3 + mt^2 + nt - 3m - n = 0$$

Then, $\quad a + b + c = \frac{-m}{l}, \quad ab + bc + ca = \frac{n}{l}$

and $\quad abc = \left(\frac{3m+n}{l}\right)$

Now, $\quad abc - (bc + ca + ab) + 3(a + b + c)$

$$= \left(\frac{3m+n}{l}\right) - \frac{n}{l} + 3\left(\frac{-m}{l}\right) = 0$$

22. ***(c)*** We have, $\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$

$$\text{IF} = e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$$

$$\therefore \quad y(1+x^2) = \int(1+x^2)\left(\frac{4x^2}{1+x^2}\right)dx + C$$

$$\Rightarrow \quad y(1+x^2) = \int 4x^2dx + C = \frac{4x^3}{3} + C$$

Since, curve passing through origin.

$$\therefore \quad c = 0$$

$$\therefore \quad y = \frac{4x^3}{3(1+x^2)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{3}\left[\frac{(1+x^2)(3x^2) - x^3(2x)}{(1+x^2)^2}\right]$$

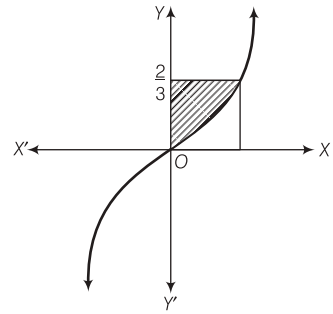
$$\Rightarrow \quad \frac{dy}{dx} = \frac{4(3+x^2)}{3(1+x^2)} \Rightarrow \frac{dy}{dx} > 0, \forall\ x \neq 0$$

$\Rightarrow f(x)$ is strictly increasing for all $x \in R$,

$\frac{dy}{dx} = 0$ at $x = 0$ and it does not change sign.

$\Rightarrow x = 0$ is the point of inflection.

23. ***(c)*** $x \to \infty, y \to \infty$ and $x \to -\infty, y \to -\infty$



Area enclosed by $y = f(x)$, X-axis and ordinate at $x = \frac{2}{3}$.

$$A = \frac{2}{3} - \frac{4}{3}\int_0^1 \frac{x^3}{1+x^2}dx$$

Put $\quad 1 + x^2 = t$

$$\Rightarrow \quad 2xdx = dt$$

$$\therefore \quad A = \frac{2}{3} - \frac{2}{3}\int_1^2 \frac{(t-1)}{t}dt = \frac{2}{3} - \frac{2}{3}\int_1^2\left(1 - \frac{1}{t}\right)dt$$

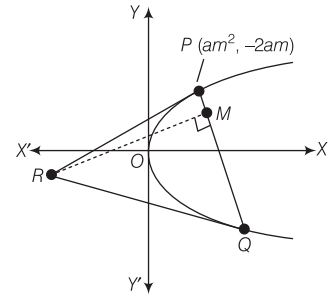
$$= \frac{2}{3} - \frac{2}{3}[t - \log t]_1^2$$

$$= \frac{2}{3} - \frac{2}{3}[(2 - \log 2) - (1 - \log 1)]$$

$$= \frac{2}{3} - \frac{2}{3}[1 - \log 2]$$

$$= \frac{2}{3}\log 2$$

24. ***(b)*** Given, normal at point $P\ (am^2, -2am)$ intersects the parabola $y^2 = 4ax$ at Q.



Let the coordinate of Q be $(am_1^2, -2am_1)$.

$$\therefore \quad m_1 = -m - \frac{2}{m}$$

Tangents at P and Q are

$$-my = x + am^2 \qquad \ldots(i)$$

and $\quad -m_1y_1 = x + am_1^2 \qquad \ldots(ii)$

Points of intersection of Eqs. (i) and (ii) is $x = -2a - am^2$

and $\quad y = -a(m + m_1) = \frac{2a}{m} \quad \left[\because m_1 = -m - \frac{2}{m}\right]$

$$\Rightarrow \quad R \equiv \left(-2a - am^2, \frac{2a}{m}\right)$$

25. ***(a)*** Let R be (x_1, y_1).

Then, PQ is the chord of contact of parabola w.r.t. R which is given by

$$yy_1 = 2a(x + x_1)$$

Length of perpendicular from R to PQ is

$$RM = \frac{y_1^2 - 4ax_1}{\sqrt{y_1^2 + 4a^2}}$$

Then, area of ΔPQR

$$= \frac{1}{2} PQ \cdot RM$$

$$= \frac{1}{2} \cdot \frac{(\sqrt{y_1^2 + 4a^2})\sqrt{(y_1^2 - 4ax_1)}(y_1^2 - 4ax_1)}{a\sqrt{y_1^2 + 4a^2}}$$

$$= \frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$$

Since, $x_1 = -2a - am^2$

and $y_1 = \frac{2a}{m}$

$$\therefore \text{ Area of } \Delta PQR = \frac{\left(\frac{4a^2}{m^2} + 8a^2 + 4a^2m^2\right)^{3/2}}{2a}$$

$$= \frac{4a^2(1 + m^2)^3}{m^3}$$

26. ***(c)*** Bag contains p red and q blue balls. Let E_i be the event of drawing a red ball in the ith draw.

Then, E'_i : event of drawing a blue ball in the ith draw.

Let E : event that P wins the game.

Then, E' : event that Q wins the game.

Since, after each draw, the ball drawn is replaced.

$$\therefore \quad P(E_i) = \frac{p}{p + q} \text{ and } P(E'_i) = \frac{q}{p + q}$$

Now,

$$P(E) = P(E_1) + P(E'_1E'_2E_3) + P(E'_1E'_2E'_3E'_4E_5) + \ldots$$

$$= P(E_1) + P(E'_1)P(E'_2)P(E_3) + P(E'_1)P(E'_2)P(E'_3)P(E'_4)P(E_5) + \ldots$$

$$= \frac{p}{p + q} + \frac{q^2}{(p + q)^2} \cdot \frac{p}{(p + q)} + \frac{q^4}{(p + q)^4} \cdot \frac{p}{p + q} + \ldots$$

$$= \frac{p}{(p + q)}\left[1 + \left(\frac{q}{p + q}\right)^2 + \left(\frac{q}{p + q}\right)^4 + \ldots\right]$$

$$= \left(\frac{p}{p + q}\right) \cdot \frac{1}{\left[1 - \left(\frac{q}{p + q}\right)^2\right]} = \frac{p(p + q)}{p^2 + 2pq} = \frac{p + q}{p + 2q}$$

$$\therefore \quad P(E') = 1 - P(E) = 1 - \frac{p + q}{p + 2q} = \frac{q}{p + 2q}$$

27. ***(b)*** Since, $P(E) = 3P(E')$

$$\therefore \quad \frac{p + q}{p + 2q} = \frac{3q}{p + 2q}$$

$$\Rightarrow \quad p + q = 3q \Rightarrow p = 2q$$

$$\Rightarrow \quad p : q = 2 : 1$$

28. A $\rightarrow$ r; B $\rightarrow$ s; C $\rightarrow$ q; D $\rightarrow$ p

A. Point $(3, \lambda, \mu)$ lies on

$$2x + y + z - 3 = 0 = x - 2y + z - 1$$

$$\Rightarrow \quad 2 \times 3 + \lambda + \mu - 3 = 0 = 3 - 2 \times \lambda + \mu - 1$$

$$\Rightarrow \quad \lambda + \mu + 3 = 0 \text{ and } 2\lambda - \mu - 2 = 0$$

$$\Rightarrow \quad \lambda + \mu = -3$$

B. If the required image is (x, y, z), then

$$\frac{x - 3}{2} = \frac{y - 5}{1} = \frac{z - 7}{1} = \frac{-2(6 + 5 + 7 + 18)}{2^2 + 1^2 + 1^2} = -12$$

$$\Rightarrow \quad x = -21, y = -7, z = -5$$

$$\therefore \quad (x, y, z) = (-21, -7, -5)$$

C. Given plane is

$$3x + 2y + 2z + 5 = 0 \qquad \ldots(i)$$

and given line is

$$\frac{x + 3}{3} = \frac{y - 2}{6} = \frac{z}{2} \qquad \ldots(ii)$$

Equation of line through point (2, 3, 4) and parallel to the line (ii) is

$$\frac{x - 2}{3} = \frac{y - 3}{6} = \frac{z - 4}{2} \qquad \ldots(iii)$$

Any point on line (iii) is $(3k + 2, 6k + 3, 2k + 4)$.

If it lies on plane (i), then $k = -1$

$\therefore$ Point on the line (iii) is $(-1, -3, 2)$.

$\therefore$ Required distance

$$= \sqrt{(2 + 1)^2 + (3 + 3)^2 + (4 + 2)^2} = 9$$

D. Any point on the line

$$\frac{x - 2}{2} = \frac{y - 2}{3} = \frac{z - 3}{4} = \lambda \text{ is } (2\lambda + 1, 3\lambda + 2, 4\lambda + 3),$$

which satisfies the line

$$\frac{x - 4}{5} = \frac{y - 1}{2} = \frac{z}{1}.$$

$$\Rightarrow \quad \frac{2\lambda + 1 - 4}{5} = \frac{3\lambda + 2 - 1}{2} = \frac{4\lambda + 3}{1}$$

$$\Rightarrow \quad \lambda = 1$$

$\therefore$ Required point is $(-1, -1, -1)$.

29. A $\rightarrow$ r, s; B $\rightarrow$ p; C $\rightarrow$ r, s; D $\rightarrow$ q

A. Since, $M^TM = 1$ and $|M| = 1$

$$\therefore \quad |M - I| = |IM - M^TM| \qquad [\because IM = M]$$

$$\Rightarrow \quad |M - I| = |(I - M^T)M|$$

$$= |(I - M)^T||M|$$

$$= |I - M|$$

$$= (-1)^3 |M - I| \quad [\because I - M \text{ is a } 3 \times 3 \text{ matrix}]$$

$$= -|M - I|$$

$$\Rightarrow 2|M - I| = 0 \Rightarrow |M - I| = 0$$

B. Given, $\begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda^2 + 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$

$$= a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e$$

On putting $\lambda = 0$, we get

To be Continued at Page 87

@ CLASS XII SYLLABUS

Rapid CONCEPT REVISION

TARGET 2017

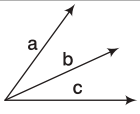
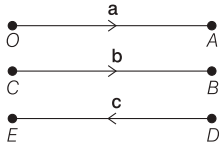
ALGEBRA OF VECTORS

The quantities, which have both magnitude and direction, are called vectors. Geometrically, a vector is represented by a directed line segment, say **AB**. Here, A is called the initial point and B is called the terminal point or tip. The magnitude or modulus of **AB** is denoted by $|\mathbf{AB}|$ or AB.

The magnitude of a vector is always a non-negative real number.

Types of Vector

On the basis of magnitude, position or their initial and terminal points, following are types of vectors

S. No.	Vectors	Definitions
1.	Zero vector	A vector whose initial and terminal points coincides, is called zero or null vector and it is denoted by **0**. Also, it can be represented by **AA**, **BB**, etc.
2.	Unit vector	A vector of unit magnitude is called a unit vector. The unit vector in the direction of a vector **a** is denoted by $\hat{\mathbf{a}}$. Also, it is defined as $\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$.
3.	Like and unlike vectors	Vectors are said to be like or unlike according as they have same or opposite sense of direction.
4.	Coinitial vectors	Two or more vectors having same initial points are called coinitial vectors. 
5.	Coterminous vectors	Vectors having same terminal points are called coterminous vectors.
6.	Negative of a vector	A vector whose magnitude is same as that of given vector but the direction is opposite to it, is called negative vector of given vector. If **AB** is a vector, then its negative vector is **BA**, i.e. **AB** = **– BA**.
7.	Free and localised vectors	Vectors whose initial points are not specified, are called free vectors, but if a vector is drawn from given specified point, then it is called localised vectors.
8.	Collinear or parallel vectors	Two or more vectors are said to be parallel, if they have the same support or parallel support. Parallel vector may have equal or unequal magnitudes and their directions may be same or opposite. 
9.	Coplanar vectors	Two or more vectors are called coplanar, if they lie on the same plane. Note that two vectors having same initial point are always coplanar but three or more vectors may or may not be coplanar.
10.	Reciprocal of a vector	A vector having the same direction as that of a given vector **a** but magnitude equal to the reciprocal of the given vector is known as the reciprocal of **a** and is denoted by $\mathbf{a}^{-1}$. Thus, if $\|\mathbf{a}\| = a$, then $\|\mathbf{a}^{-1}\| = \frac{1}{a}$.

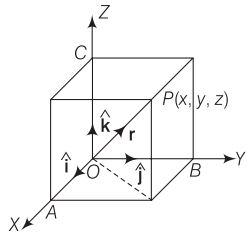
S. No.	Vectors	Definitions
11.	Equal vectors	Two vectors **a** and **b** are said to be equal, if they have same magnitude and direction, regardless of the positions of their initial points. We write it as $\mathbf{a} = \mathbf{b}$.
12.	Position vector	The vector **OP**, which represents the position of the point P with respect to a fixed point O (called origin), is called position vector of the point P. If (x, y, z) is coordinates of point P, then $\|\mathbf{OP}\| = \sqrt{x^2 + y^2 + z^2}$.

Components of Vector

If $P(x, y, z)$ is any point in space, then

$\mathbf{OP} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

Here, x, y and z are called the scalar components of **OP** and $x\hat{\mathbf{i}}, y\hat{\mathbf{j}}, z\hat{\mathbf{k}}$ are called the vector components of **r** along the respective axes.



Direction Cosines

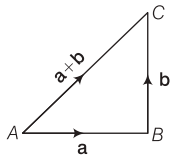
The angles, which the vector makes with the positive directions of X, Y and Z-axes, are called the direction angles and their cosines are called the direction cosines. If component form of vector **a** is $\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and α, β, γ are the angles made by the vector **a** with X, Y and Z-axes, then

$$\cos\alpha = \frac{x}{|\mathbf{a}|},\ \cos\beta = \frac{y}{|\mathbf{a}|} \text{ and } \cos\gamma = \frac{z}{|\mathbf{a}|}$$

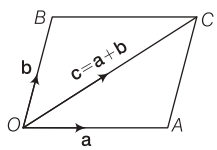
Note $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Addition of Vectors

(i) **By triangle law** If two vectors, say **a** and **b** are represented by two sides of a triangle taken in the same order, then their sum $(\mathbf{a} + \mathbf{b})$ can be represented by third side of the triangle taken in the reverse order.



(ii) **By parallelogram law** If two vectors, say **a** and **b** are represented by two adjacent sides of a parallelogram, then their sum $(\mathbf{a} + \mathbf{b})$ can be represented by the diagonal of the parallelogram whose initial point is same as the initial point of the given vectors.



Addition in Component Form

Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$

and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Then, $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{\mathbf{i}} + (a_2 + b_2)\hat{\mathbf{j}} + (a_3 + b_3)\hat{\mathbf{k}}$

Properties of Vector Addition

Following are the properties of vector addition

(i) Vector addition is commutative, i.e. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

(ii) Vector addition is associative, i.e. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

(iii) $\mathbf{0}$ (null vector) is the additive identity as $\mathbf{a} + \mathbf{0} = \mathbf{a} = \mathbf{0} + \mathbf{a}$.

(iv) $(-\mathbf{a})$ is additive inverse of **a**, as $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$, i.e. negative of vector is its additive inverse.

Subtraction of Vectors

If **a** and **b** are two vectors, then $\mathbf{a} - \mathbf{b}$ is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

Note Let A and B be two given points with position vectors **OA** and **OB**, then **AB** = **OB** – **OA**.

Subtraction in Component Form

Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$

and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Then, $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\hat{\mathbf{i}} + (a_2 - b_2)\hat{\mathbf{j}} + (a_3 - b_3)\hat{\mathbf{k}}$

Properties of Vector Subtraction

(i) $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$

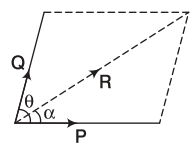
(ii) $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$

REMEMBER

- **Triangle Inequality**

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$
$$|\mathbf{a} + \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$$
$$|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$
$$|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$$

- The internal bisector of the angle between any two vectors is along the vector sum of the corresponding unit vectors.
- The external bisector of the angle between any two vectors is along the vector difference of the corresponding unit vectors.
- If **R** represents the resultant of two forces **P** and **Q**, then
  (a) $\mathbf{R} = \mathbf{P} + \mathbf{Q}$
  (b) $|\mathbf{R}| = R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$, where $|\mathbf{P}| = P$ and $|\mathbf{Q}| = Q$
  (c) $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$



Scalar Multiplication of a Vector

If **a** is a vector and m is a scalar, then $m\mathbf{a}$ represents the vector whose direction is same, when $m > 0$ or opposite, when $m < 0$ and magnitude is m times of $|\mathbf{a}|$.

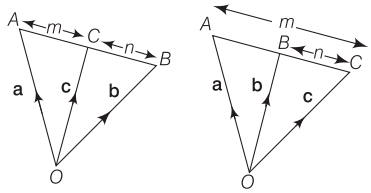
Properties of Scalar Multiplication of Vectors

If $\mathbf{a}$, $\mathbf{b}$ are vectors and m,n are scalars, then following are true.

(i) $m(\mathbf{a}) = \mathbf{a}(m)$
(ii) $m(n\mathbf{a}) = n(m\mathbf{a}) = mn(\mathbf{a})$
(iii) $(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$
(iv) $m(\mathbf{a}+\mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

Section Formulae

If $\mathbf{a}$ and $\mathbf{b}$ are position vectors of two points A and B, then the position vector $\mathbf{c}$ of the point C, which divides AB in the ratio $m : n$, is



$$\mathbf{c} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n} \quad \text{[internal division]}$$

and $$\mathbf{c} = \frac{m\mathbf{b} - n\mathbf{a}}{m-n} \quad \text{[external division]}$$

Linear Combination

A vector $\mathbf{r}$ is said to be linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$, if there exist scalars $k_1, k_2, k_3, \ldots, k_n$ such that $\mathbf{r} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_3\mathbf{a}_3 + \ldots + k_n\mathbf{a}_n$.

Linearly Dependent

A system of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$ is said to be linearly dependent, if there exist scalars $k_1, k_2, k_3, \ldots, k_n$, not all zero, such that

$$k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_3\mathbf{a}_3 + \ldots + k_n\mathbf{a}_n = 0$$

Linearly Independent

A system of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is said to be linearly independent, if

$$k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_3\mathbf{a}_3 + \ldots + k_n\mathbf{a}_n = 0$$
$$\Rightarrow \quad k_1 = k_2 = k_3 = \ldots = k_n = 0$$

Note The vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ will be linearly dependent iff

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Properties of Parallel Vectors

(i) If $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
(ii) If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero, non-parallel vectors, then $k_1\mathbf{a} + k_2\mathbf{b} = 0 \Rightarrow k_1 = k_2 = 0$,
i.e. two non-zero, non-collinear vectors are linearly independent.
(iii) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{a} \parallel \mathbf{b}$, then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

Test of Collinearity of Three Points

Three points A, B and C with position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively are collinear if and only if there exist scalars x, y and z, not all zero simultaneously, such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$, where $x + y + z = 0$.

Note Three points with position vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}}$; $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}$ are collinear iff

$$\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0$$

Test of Coplanarity of Three Vectors

Three vectors are coplanar if one of them is expressible as a linear combination of other two. In other words, let $\mathbf{a}$ and $\mathbf{b}$ be two given non-zero, non-collinear vectors. Then, any vector $\mathbf{r}$ will be coplanar with $\mathbf{a}$ and $\mathbf{b}$, if it can be uniquely expressed as $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$, for some scalars x and y.

Test of Coplanarity of Four Points

Four points with position vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ and $\mathbf{d} = d_1\hat{\mathbf{i}} + d_2\hat{\mathbf{j}} + d_3\hat{\mathbf{k}}$ will be coplanar iff

$$\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0$$

REMEMBER

- Any two collinear vectors are linearly dependent.
- Any three non-coplanar vectors are linearly independent.
- Any three coplanar vectors are linearly dependent.
- Any four vectors in 3-dimensional space are linearly dependent.

DO YOU Know?

Fill the empty circles with appropriate number.

① ② ③
④ ⑤ ④ ⑥ ⑦
② ① ⓪ ⑩ ④ ⑥ ③
⑥ ⑦ ? ⑧ ⑨
② ⑥ ⑧

MASTER STROKES

1. If $ABCDEF$ is a regular hexagon, then $\mathbf{AD} + \mathbf{EB} + \mathbf{FC}$ equals
(a) $2\,\mathbf{AB}$ (b) 0 (c) $3\,\mathbf{AB}$ (d) $4\,\mathbf{AB}$

2. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are the position vectors of the points A, B, C and D such that no three of them are collinear and $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$, then $ABCD$ is a
(a) rhombus (b) rectangle
(c) square (d) parallelogram

3. $ABCDEF$ is a regular hexagon with centre at the origin such that $\mathbf{AD} + \mathbf{EB} + \mathbf{FC} = \lambda\,\mathbf{ED}$. Then, λ equals
(a) 2 (b) 4 (c) 6 (d) 3

4. If $ABCD$ is a parallelogram with AC and BD as diagonals. Then, $\mathbf{AC} - \mathbf{BD}$ is equal to
(a) $4\,\mathbf{AB}$ (b) $3\,\mathbf{AB}$ (c) $2\,\mathbf{AB}$ (d) $\mathbf{AB}$

5. If $OACB$ is a parallelogram with $\mathbf{OC} = \mathbf{a}$ and $\mathbf{AB} = \mathbf{b}$, then $\mathbf{OA}$ is equal to
(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$ (c) $\frac{1}{2}(\mathbf{b} - \mathbf{a})$ (d) $\frac{1}{2}(\mathbf{a} - \mathbf{b})$

6. If G is the intersection of diagonals of a parallelogram $ABCD$ and O is any point, then $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}$ equals
(a) $2\,\mathbf{OG}$ (b) $4\,\mathbf{OG}$ (c) $5\,\mathbf{OG}$ (d) $3\,\mathbf{OG}$

7. Let G be the centroid of ΔABC. If $\mathbf{AB} = \mathbf{a}$ and $\mathbf{AC} = \mathbf{b}$, then $\mathbf{AG}$, in terms of $\mathbf{a}$ and $\mathbf{b}$, is
(a) $\frac{2}{3}(\mathbf{a} + \mathbf{b})$ (b) $\frac{1}{6}(\mathbf{a} + \mathbf{b})$ (c) $\frac{1}{3}(\mathbf{a} + \mathbf{b})$ (d) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

8. The position vectors of the points A, B and C are $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, respectively. These points
(a) form an isosceles triangle (b) form a right triangle
(c) are collinear (d) form a scalene triangle

9. If the points with position vectors $20\hat{\mathbf{i}} + p\hat{\mathbf{j}}$, $5\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $10\hat{\mathbf{i}} - 13\hat{\mathbf{j}}$ are collinear, then p is equal to
(a) 7 (b) -37 (c) -7 (d) 37

10. If the position vector of a point A is $\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a}$ divides AB in the ratio $2 : 3$, then the position vector of B is
(a) $2\mathbf{a} - \mathbf{b}$ (b) $\mathbf{b} - 2\mathbf{a}$ (c) $\mathbf{a} - 3\mathbf{b}$ (d) $\mathbf{b}$

11. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-zero vectors, no two of which are collinear and the vector $\mathbf{a} + \mathbf{b}$ is collinear with $\mathbf{c}$, $\mathbf{b} + \mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is equal to
(a) $\mathbf{a}$ (b) $\mathbf{b}$ (c) $\mathbf{c}$ (d) None of these

12. If points $A(60\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$, $B(40\hat{\mathbf{i}} - 8\hat{\mathbf{j}})$ and $C(a\hat{\mathbf{i}} - 52\hat{\mathbf{j}})$ are collinear, then a is equal to
(a) 40 (b) -40 (c) 20 (d) -20

13. Let $\mathbf{OA} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{OB} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$. Then, vector $\mathbf{OC}$ bisecting $\angle AOB$ and C being a point on the line AB, is
(a) $4(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ (b) $2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$
(c) $\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ (d) None of these

14. If the vector $-\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ bisects the angle between the vector $\mathbf{c}$ and the vector $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$, then the unit vector in the direction of $\mathbf{c}$ is
(a) $\frac{1}{15}(11\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ (b) $-\frac{1}{15}(11\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$
(c) $-\frac{1}{15}(11\hat{\mathbf{i}} + 10\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (d) $-\frac{1}{15}(11\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

15. If $\mathbf{r} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$, $\mathbf{a} = 2\hat{\mathbf{j}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{c} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$, then
(a) x, y, z are in AP (b) x, y, z are in GP
(c) x, y, z are in HP (d) $y, \frac{x}{2}, z$ are in AP

16. Let $\mathbf{AB} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{AC} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and a point P on the line segment BC is equidistant from AB and AC, then $\mathbf{AP}$ is equal to
(a) $2\hat{\mathbf{i}} - \hat{\mathbf{k}}$ (b) $\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}} + \hat{\mathbf{k}}$ (d) None of these

17. The vector $\mathbf{c}$, directed along the internal bisector of the angle between the vectors $\mathbf{a} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\mathbf{b} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ with $|\mathbf{c}| = 5\sqrt{6}$, is
(a) $\frac{5}{3}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ (b) $\frac{5}{3}(5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$
(c) $\frac{5}{3}(\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ (d) $\frac{5}{3}(-5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

18. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-zero vectors which are pairwise non-collinear. If $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}$ is equal to
(a) $\mathbf{c}$ (b) 0 (c) $\mathbf{a} + \mathbf{c}$ (d) $\mathbf{a}$

19. If the vectors $\mathbf{AB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$ and $\mathbf{AC} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ are the sides of a ΔABC, then the length of the median through A is
(a) $\sqrt{18}$ (b) $\sqrt{72}$ (c) $\sqrt{33}$ (d) $\sqrt{45}$

20. Suppose that $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ non-coplanar vectors in R^3. Let the components of a vector $\mathbf{s}$ along $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ be 4, 3 and 5, respectively. If the components of this vector $\mathbf{s}$ along $-\mathbf{p} + \mathbf{q} + \mathbf{r}$, $\mathbf{p} - \mathbf{q} + \mathbf{r}$ and $-\mathbf{p} - \mathbf{q} + \mathbf{r}$ are x, y and z respectively, then the value of $2x + y + z$ is
(a) 7 (b) 8 (c) 9 (d) 6

Answers

1. (d)	2. (d)	3. (b)	4. (c)	5. (b)
6. (b)	7. (c)	8. (c)	9. (b)	10. (c)
11. (d)	12. (b)	13. (b)	14. (d)	15. (d)
16. (c)	17. (a)	18. (a)	19. (c)	20. (c)

PRODUCT OF TWO VECTORS

Product of two vectors can be scalar or vector.

Scalar (or Dot) Product of Two Vectors

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as product of magnitude $|\mathbf{a}|$ and $|\mathbf{b}|$ and cosine of angle between them, i.e. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta, (0 \le \theta \le \pi)$

Properties of Scalar Product

(i) $\mathbf{a}\cdot\mathbf{b} > 0$, when θ is acute and $\mathbf{a}\cdot\mathbf{b} < 0$, when θ is obtuse.

(ii) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = a^2$

(iii) Scalar product is commutative, i.e. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

(iv) Scalar product is distributive over addition.
i.e. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

(v) If $\mathbf{a}\cdot\mathbf{b} = 0$, then either $\mathbf{a} = 0$ or $\mathbf{b} = 0$ or $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

(vi) $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$; $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{i}} = 0$;
$\hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{j}} = 0$; $\hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = 0$

(vii) The angle between $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|},\ 0 \le \phi \le \pi$$

(viii) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Then, $\mathbf{a}\cdot\mathbf{b} = (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})\cdot(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}})$

$\Rightarrow \mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

In particular, $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$

SPEED BOOSTER

For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

- Maximum value of $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$
- Minimum value of $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$
- $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| \Rightarrow \mathbf{a} \| \mathbf{b}$
- $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \Rightarrow \mathbf{a} \perp \mathbf{b}$
- $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Rightarrow \mathbf{a} \perp \mathbf{b}$

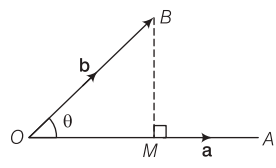
Projection of Vector

Projection of vector $\mathbf{a}$ on vector $\mathbf{b} = |\mathbf{a}|\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

Projection of vector $\mathbf{b}$ on vector $\mathbf{a} = |\mathbf{b}|\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Resolving a Given Vector in the Direction of Two Given Perpendicular Vectors

The components of $\mathbf{b}$ along and perpendicular to $\mathbf{a}$ are $\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ and $\mathbf{b} - \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$, respectively.



Vector (or Cross) Product of Two Vectors

If $\mathbf{a}$, $\mathbf{b}$ are two vectors and θ is the angle between them, then $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, and $\hat{\mathbf{n}}$ forms a right handed system.

Properties of Vector Product

(i) Vector product is not commutative,
i.e. $\mathbf{a}\times\mathbf{b} \ne \mathbf{b}\times\mathbf{a}$, infact $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.

(ii) Vector product is not associative,
i.e. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \ne \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

(iii) Vector product is distributive over addition and subtraction, i.e. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}-\mathbf{c}) = \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c}$

(iv) $\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = 0$

(v) $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}}$, $\hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$, also $\hat{\mathbf{j}}\times\hat{\mathbf{i}} = -\hat{\mathbf{k}}$,
$\hat{\mathbf{i}}\times\hat{\mathbf{k}} = -\hat{\mathbf{j}}$, $\hat{\mathbf{k}}\times\hat{\mathbf{j}} = -\hat{\mathbf{i}}$

(vi) If $\hat{\mathbf{a}} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(vii) Unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ is given by $\hat{\mathbf{n}} = \pm\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$.

(viii) A vector of magnitude k and perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ is $\pm\dfrac{k(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$.

(ix) Geometrically, $|\mathbf{a}\times\mathbf{b}|$ is the area of the parallelogram, whose two adjacent sides are represented by $\mathbf{a}$ and $\mathbf{b}$.

(x) The area of ΔABC is

$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| \text{ or } \frac{1}{2}|\mathbf{BC}\times\mathbf{BA}|$$

or $\dfrac{1}{2}|\mathbf{CB}\times\mathbf{CA}|$

(xi) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of three points A, B and C, then the area of ΔABC

$$= \frac{1}{2}[\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}]$$

(xii) The area of any quadrilateral whose diagonal vectors are $\mathbf{d}_1$ and $\mathbf{d}_2$, is given by $\dfrac{1}{2}|\mathbf{d}_1\times\mathbf{d}_2|$.

(xiii) If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\cdot|\mathbf{b}|}$

(xiv) $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} \,||\, \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

(xv) If $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are three non-coplanar vectors, then any vector $\mathbf{r}$ can be written as

$$\mathbf{r} = \frac{(\mathbf{r} \cdot \vec{\alpha})}{|\vec{\alpha}|^2}\vec{\alpha} + \frac{(\mathbf{r} \cdot \vec{\beta})}{|\vec{\beta}|^2}\beta + \frac{(\mathbf{r} \cdot \vec{\gamma})}{|\vec{\gamma}|^2}\vec{\gamma}$$

REMEMBER

- Three points A, B and C are collinear iff $\mathbf{AB} \times \mathbf{AC} = 0$.
  Note that, if $\mathbf{AB} = a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}}$, $\mathbf{AC} = a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then A, B and C will be collinear.
- Three points A, B and C with position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively are collinear, if $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a}) = 0$.

MASTER STROKES

1. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are any four vectors, then $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ is a vector
(a) perpendicular to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$
(b) along the line of intersection of the plane determined by $\mathbf{a}, \mathbf{b}$ and the other determined by $\mathbf{c}$ and $\mathbf{d}$
(c) equally inclined to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$
(d) None of the above

2. The acute angle between the medians drawn through the acute angles of an isosceles right angled triangle, is
(a) $\cos^{-1}\left(\frac{2}{3}\right)$ *(b)* $\cos^{-1}\left(\frac{3}{4}\right)$ *(c)* $\cos^{-1}\left(\frac{4}{5}\right)$ *(d)* $\cos^{-1}\left(\frac{5}{6}\right)$

3. Let Π_1 and Π_2 be the two planes determined by the pairs of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$, respectively. If the planes Π_1 and Π_2 are parallel, then
(a) $(\mathbf{a} \times \mathbf{c}) \times (\mathbf{b} \times \mathbf{d}) = 0$ *(b)* $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{a} \times \mathbf{d}) = 0$
(c) $(\mathbf{c} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{d}) = 0$ *(d)* $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$

4. Let three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be such that $\mathbf{a} \times \mathbf{b} = 3(\mathbf{a} \times \mathbf{c})$. Also, $|\mathbf{a}| = |\mathbf{b}| = 1$, $|\mathbf{c}| = \frac{1}{3}$. If the angle between $\mathbf{b}$ and $\mathbf{c}$ is 60°, then
(a) $\mathbf{b} = 3\mathbf{c} + \mathbf{a}$ *(b)* $\mathbf{b} = 3\mathbf{a} - \mathbf{c}$
(c) $\mathbf{b} = 3\mathbf{c} + 2\mathbf{a}$ *(d)* $\mathbf{b} = 3\mathbf{c} - 2\mathbf{a}$

5. Let the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ be such that $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$ respectively, then the angle between P_1 and P_2 is
(a) 0 *(b)* $\frac{\pi}{4}$ *(c)* $\frac{\pi}{3}$ *(d)* $\frac{\pi}{2}$

6. If $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are the unit vectors, then $|\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 + |\hat{\mathbf{b}} - \hat{\mathbf{c}}|^2 + |\hat{\mathbf{c}} - \hat{\mathbf{a}}|^2$ does not excess
(a) 4 *(b)* 9 *(c)* 8 *(d)* 6

7. If $\mathbf{b}$ and $\mathbf{c}$ are any two non-collinear vectors and $\mathbf{a}$ is any vector, then $(\mathbf{a} \cdot \mathbf{b})\mathbf{b} + (\mathbf{a} \cdot \mathbf{c})\mathbf{c} + \frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{|\mathbf{b} \times \mathbf{c}|^2}(\mathbf{b} \times \mathbf{c})$ is equal to
(a) $\mathbf{a}$ *(b)* $\mathbf{b}$
(c) $\mathbf{c}$ *(d)* None of these

8. If $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear vectors such that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and $\mathbf{a} - \mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, then the value of $\left\{\frac{\mathbf{a}}{|\mathbf{a}|^2} - \frac{\mathbf{b}}{|\mathbf{b}|^2}\right\}^2$ is equal to
(a) $\frac{1}{24}$ *(b)* $\frac{5}{72}$ *(c)* $\frac{7}{72}$ *(d)* $\frac{7}{48}$

9. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three mutually perpendicular vectors, then the projection of the vector $l\frac{\mathbf{a}}{|\mathbf{a}|} + m\frac{\mathbf{b}}{|\mathbf{b}|} + n\frac{(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$ along the angle bisector of the vectors $\mathbf{a}$ and $\mathbf{b}$, is
(a) $\frac{l^2 + m^2}{\sqrt{l^2 + m^2 + n^2}}$ *(b)* $\sqrt{l^2 + m^2 + n^2}$
(c) $\frac{\sqrt{l^2 + m^2}}{\sqrt{l^2 + m^2 + n^2}}$ *(d)* $\frac{l + m}{\sqrt{2}}$

10. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{r}$ be a variable vector such that $\mathbf{r} \cdot \hat{\mathbf{i}}, \mathbf{r} \cdot \hat{\mathbf{j}}$ and $\mathbf{r} \cdot \hat{\mathbf{k}}$ are positive integers. If $\mathbf{r} \cdot \mathbf{a} \le 12$, then the total number of such vectors is
(a) ${}^{12}C_9 - 1$ *(b)* ${}^{12}C_3$
(c) ${}^{12}C_8$ *(d)* None of these

11. Let us define the length of a vector $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ as $|a| + |b| + |c|$. This definition coincides with the usual definition of the length of a vector $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ iff,
(a) $a = b = c = 0$
(b) any two of a, b and c are zero
(c) any one of a, b and c is zero
(d) $a + b + c = 0$

12. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors satisfying $\mathbf{a} \times \mathbf{b} = 2(\mathbf{a} \times \mathbf{c})$, $|\mathbf{a}| = |\mathbf{c}| = 1$, $|\mathbf{b}| = 4$ and $|\mathbf{b} \times \mathbf{c}| = \sqrt{15}$. If $\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$, then λ equals
(a) 1 *(b)* –1 *(c)* 2 *(d)* –4

13. The vector $\mathbf{a}(x) = \cos x\hat{\mathbf{i}} + \sin x\hat{\mathbf{j}}$ and $\mathbf{b}(x) = x\hat{\mathbf{i}} + \sin x\hat{\mathbf{j}}$ are collinear for
(a) unique value of x, $0 < x < \frac{\pi}{6}$
(b) unique value of x, $\frac{\pi}{6} < x < \frac{\pi}{3}$
(c) no value of x
(d) infinitely many values of x, $0 < x < \frac{\pi}{2}$

14. If $\mathbf{a} = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$ and $\mathbf{b} = 6\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ are two vectors and $\mathbf{c}$ is a vector such that $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, then $|\mathbf{a}| : |\mathbf{b}| : |\mathbf{c}|$ is equal to
(a) $\sqrt{34} : \sqrt{45} : \sqrt{39}$ *(b)* $\sqrt{34} : \sqrt{45} : 39$
(c) 34 : 39 : 45 *(d)* 39 : 35 : 34

15. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three vectors such that $|\mathbf{u}| = 1$, $|\mathbf{v}| = 2$, $|\mathbf{w}| = 3$. If the projection of $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{w}$ are perpendicular to each other, then $|\mathbf{u} - \mathbf{v} + \mathbf{w}|$ equals

(a) 14 *(b)* $\sqrt{7}$ *(c)* $\sqrt{14}$ *(d)* 2

16. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero and non-coplanar vectors and $\mathbf{b}_1 = \mathbf{b} - \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$, $\mathbf{b}_2 = \mathbf{b} + \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$

and $\mathbf{c}_1 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} - \frac{(\mathbf{c}\cdot\mathbf{b})}{|\mathbf{b}|^2}\mathbf{b}$,

$\mathbf{c}_2 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} - \frac{(\mathbf{c}\cdot\mathbf{b}_1)}{|\mathbf{b}_1|^2}\mathbf{b}_1$

$\mathbf{c}_3 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} - \mathbf{c} - \frac{(\mathbf{c}\cdot\mathbf{b}_2)}{|\mathbf{b}_2|^2}\mathbf{b}_2$

$\mathbf{c}_4 = \mathbf{a} - \frac{(\mathbf{c}\cdot\mathbf{a})}{|\mathbf{a}|^2}\mathbf{a}$

Then, which one of the following is a set of mutually orthogonal vectors?

(a) $\{\mathbf{a}, \mathbf{b}_1, \mathbf{c}_1\}$ *(b)* $\{\mathbf{a}, \mathbf{b}_1, \mathbf{c}_2\}$
(c) $\{\mathbf{a}, \mathbf{b}_2, \mathbf{c}_3\}$ *(d)* $\{\mathbf{a}, \mathbf{b}_2, \mathbf{c}_4\}$

17. Let $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$. If a vector coplanar to $\mathbf{a}$ and $\mathbf{b}$ has a projection along $\mathbf{c}$ of magnitude $1/\sqrt{3}$, then the vector is

(a) $4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ *(b)* $4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ *(d)* None of these

18. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ and $\hat{\mathbf{d}}$ are the unit vectors such that $(\hat{\mathbf{a}} \times \hat{\mathbf{b}}) \cdot (\hat{\mathbf{c}} \times \hat{\mathbf{d}}) = 1$ and $\hat{\mathbf{a}} \cdot \hat{\mathbf{c}} = \frac{1}{2}$, then

(a) $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are non-coplanar
(b) $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ and $\hat{\mathbf{d}}$ are non-coplanar
(c) $\hat{\mathbf{b}}$, $\hat{\mathbf{d}}$ are non-parallel
(d) $\hat{\mathbf{a}}$, $\hat{\mathbf{d}}$ are parallel and $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ are parallel

19. If G is the centroid of ΔABC, then $\frac{AG^2 + BG^2 + CG^2}{AB^2 + BC^2 + CA^2}$ is equal to

(a) 1/4 *(b)* 1/3 *(c)* 2/3 *(d)* 4/9

20. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{\mathbf{i}}$, $-\hat{\mathbf{j}}$, $4\hat{\mathbf{i}}$, $3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, respectively. Then, the quadrilateral $PQRS$ must be a

(a) parallelogram, which is neither rhombus nor rectangle
(b) square
(c) rectangle, but not a square
(d) rhombus, but not a square

21. Let non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy $\mathbf{a}\cdot\mathbf{b} = 0$, $(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} + \mathbf{c}) = 0$ and $2|\mathbf{b} + \mathbf{c}| = |\mathbf{b} - \mathbf{a}|$. If $\mathbf{a} = \mu\mathbf{b} + 4\mathbf{c}$, then the possible values of μ are

(a) 0, 4 *(b)* 0, 5 *(c)* 0, 6 *(d)* 4, 5

22. If $\mathbf{a} = \hat{\mathbf{j}} + \sqrt{3}\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{j}} + \sqrt{3}\hat{\mathbf{k}}$ and $\mathbf{c} = 2\sqrt{3}\hat{\mathbf{k}}$ form a triangle, then the internal angle of the triangle between $\mathbf{a}$ and $\mathbf{b}$ is

(a) $\frac{\pi}{6}$ *(b)* $\frac{2\pi}{3}$ *(c)* $\frac{\pi}{3}$ *(d)* $\frac{\pi}{2}$

23. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$ be three vectors. A vector $\mathbf{r}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$, whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is given by

(a) $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ *(b)* $-3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$
(c) $3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ *(d)* $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

24. If $\mathbf{a}$ and $\mathbf{b}$ are vectors such that $|\mathbf{a} + \mathbf{b}| = \sqrt{29}$ and $\mathbf{a} \times (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \times \mathbf{b}$, then the possible value of $(\mathbf{a} + \mathbf{b})(-7\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ is

(a) 0 *(b)* 3 *(c)* 4 *(d)* 8

25. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are the unit vectors satisfying $|\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 + |\hat{\mathbf{b}} - \hat{\mathbf{c}}|^2 + |\hat{\mathbf{c}} - \hat{\mathbf{a}}|^2 = 9$, then $|2\hat{\mathbf{a}} + 5\hat{\mathbf{b}} + 5\hat{\mathbf{c}}|$ is equal to

(a) 0 *(b)* 1 *(c)* 2 *(d)* 3

26. Let $A_1, A_2, \ldots, A_n (n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\mathbf{a}_k$ be the position vector of the points $A_k, k = 1, 2, \ldots, n$. If $\left|\sum_{k=1}^{n-1}(\mathbf{a}_k \times \mathbf{a}_{k+1})\right| = \left|\sum_{k=1}^{n-1}(\mathbf{a}_k \cdot \mathbf{a}_{k+1})\right|$, then the minimum value of n is

(a) 1 *(b)* 2 *(c)* 8 *(d)* 9

27. Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\hat{\mathbf{a}} \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \times \hat{\mathbf{c}} = p\hat{\mathbf{a}} + q\hat{\mathbf{b}} + r\hat{\mathbf{c}}$, where p, q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is

(a) 2 *(b)* 4 *(c)* 6 *(d)* 8

28. In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}}$ on $\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{j}}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then the possible value(s) of $|\alpha|$ is/are

(a) 1, 2 *(b)* 3, 4 *(c)* 4, 5 *(d)* 3

29. In R^2, let $\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}}$ and $\beta\hat{\mathbf{i}} + (1 - \beta)\hat{\mathbf{j}}$ be the position vectors of X, Y and Z respectively w.r.t. the origin O. If the distance of Z from the bisectors of the acute angle of OX with OY is $\frac{3}{\sqrt{2}}$, then the possible values of $|\beta|$ are

(a) 8, 9 *(b)* 7, 8, 9 *(c)* 1, 2 *(d)* 3, 4

30. Let $ABCD$ be a tetrahedron with $AB = 41$, $BC = 36$, $CA = 7$, $DA = 18$, $DB = 27$ and $DC = 13$. Then, the square of the distance between the mid-points of AB and CD equals

(a) 1096 *(b)* 548 *(c)* 274 *(d)* 137

Answers

1. (b)	**2.** (c)	**3.** (d)	**4.** (a)	**5.** (a)
6. (b)	**7.** (a)	**8.** (c)	**9.** (d)	**10.** (b)
11. (b)	**12.** (d)	**13.** (b)	**14.** (b)	**15.** (c)
16. (b)	**17.** (a)	**18.** (c)	**19.** (b)	**20.** (a)
21. (b)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (d)
26. (d)	**27.** (b)	**28.** (a)	**29.** (c)	**30.** (d)

PRODUCT OF THREE VECTORS

Product of three vectors can be scalar triple product or vector triple product.

Scalar Triple Product

The scalar triple product of three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is defined by $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = |\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\theta\cos\phi$, where θ is angle between $\mathbf{a}$ and $\mathbf{b}$, while ϕ is angle between vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}$. Scalar triple product is also defined as box product and represented as $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Properties of Scalar Triple Product

(i) Geometrically, scalar triple product represents the volume of the parallelopiped, whose three coterminous edges are represented by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ i.e. $V = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

(ii) Dot and cross can interchange their position in scalar triple product i.e. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.
Also, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

(iii) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})$, i.e. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$

(iv) $[k\,\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = k[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (where k is scalar)

(v) $[\mathbf{a}+\mathbf{b}\ \mathbf{c}\ \mathbf{d}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$

(vi) $[\mathbf{a}-\mathbf{b}\ \ \mathbf{b}-\mathbf{c}\ \ \mathbf{c}-\mathbf{a}] = 0$ and
$[\mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(vii) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

(viii) If any two of three vectors are same, then the vectors are coplanar, i.e. $[\mathbf{a}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{b}\ \mathbf{b}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{c}\ \mathbf{b}] = 0$.

(ix) Scalar product of three vectors, two of which are equal and parallel, is 0 i.e. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

(x) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(xi) In general, if $\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n}$, $\mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n}$ and $\mathbf{c} = c_1\mathbf{l} + c_2\mathbf{m} + c_3\mathbf{n}$, then
$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{l}\ \mathbf{m}\ \mathbf{n}]$
where, $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ are non-coplanar vectors.

(xii) If $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] > 0$ for right handed system and $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] < 0$ for left handed system.

(xiii) The volume of the tetrahedron $OABC$, with O as origin and the position vectors of A, B and C being $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, is given by $\frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

(xiv) The position vector of centroid of a tetrahedron, if the position vectors of its angular vertices are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, is given by $\frac{1}{4}[\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}]$.

(xv) The angle between any two plane faces of a regular tetrahedron is $\cos^{-1}\left(\frac{1}{3}\right)$.

(xvi) The angle between any edge and a face not containing the angle is $\cos^{-1}\sqrt{\frac{1}{3}}$ (For regular tetrahedron).

REMEMBER

Lagrange's Identity

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are four vectors, then $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$.

Particularly, $(\mathbf{a}\times\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})^2$

Vector Triple Product

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be any three vectors, then the expression $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a vector and is called vector triple product. Here, the vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is perpendicular to the $\mathbf{a}$ and lie in the plane of $\mathbf{b}$ and $\mathbf{c}$, i.e. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ can be written as $x\mathbf{b} + y\mathbf{c}$.

Note (i) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$
(ii) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a}$

Properties of Vector Triple Product

(i) In general, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. But if, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, then some or all $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are zero vectors, or $\mathbf{a}$ and $\mathbf{c}$ are collinear.

(ii) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]\,\mathbf{c} - [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\,\mathbf{d}$
$= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d}$

(iii) $\hat{\mathbf{i}}\times(\mathbf{a}\times\hat{\mathbf{i}}) + \hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{j}}) + \hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}}) = 2\mathbf{a}$

(iv) $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

Reciprocal System of Vectors

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are two sets of non-coplanar vectors such that $\mathbf{a}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}' = 1$, then the two systems are called reciprocal system of vectors.

Also, $\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$, $\mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$, $\mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

Properties of Reciprocal System of Vectors

(i) $\mathbf{a}\cdot\mathbf{b}' = \mathbf{a}\cdot\mathbf{c}' = \mathbf{b}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{c}' = \mathbf{c}\cdot\mathbf{a}' = \mathbf{c}\cdot\mathbf{b}' = 0$

(ii) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\cdot[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = 1$

(iii) The orthogonal triad of vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ is self reciprocal.

(iv) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar iff $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are non-coplanar.

REMEMBER

- Solving a vector equation means determining an unknown vector (or a number of vectors satisfying the given conditions).
- Generally, to solve vector equations we express the unknown as the linear combination of three non-coplanar vectors, for e.g. $\mathbf{r} = x\,\mathbf{a} + y\,\mathbf{b} + z\,(\mathbf{a}\times\mathbf{b})$; as $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ are non-coplanar and find x, y and z using given conditions.

MASTER STROKES

1. If $\hat{\mathbf{r}}$ is a unit vector such that $\hat{\mathbf{r}} = x(\mathbf{b}\times\mathbf{c}) + y(\mathbf{c}\times\mathbf{a}) + z(\mathbf{a}\times\mathbf{b})$, then $|(\hat{\mathbf{r}}\cdot\mathbf{a})(\mathbf{b}\times\mathbf{c}) + (\hat{\mathbf{r}}\cdot\mathbf{b})(\mathbf{c}\times\mathbf{a}) + (\hat{\mathbf{r}}\cdot\mathbf{c})(\mathbf{a}\times\mathbf{b})|$ is equal to
(a) $|[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]|$ *(b)* 1
(c) $3\,|[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]|$ *(d)* 0

2. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors such that $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = 2$. If $\mathbf{r} = l(\mathbf{b}\times\mathbf{c}) + m(\mathbf{c}\times\mathbf{a}) + n(\mathbf{a}\times\mathbf{b})$ be perpendicular to $\mathbf{a} + \mathbf{b} + \mathbf{c}$, then the value of $l + m + n$ is
(a) 2 *(b)* 1
(c) 0 *(d)* None of these

3. If $\hat{\mathbf{b}}$ is a unit vector, then $(\mathbf{a}\cdot\hat{\mathbf{b}})\,\hat{\mathbf{b}} + \hat{\mathbf{b}}\times(\mathbf{a}\times\hat{\mathbf{b}})$ is equal to
(a) $|\mathbf{a}|^2\hat{\mathbf{b}}$ *(b)* $(\mathbf{a}\cdot\hat{\mathbf{b}})\,\mathbf{a}$ *(c)* $\mathbf{a}$ *(d)* $(\mathbf{a}\cdot\hat{\mathbf{b}})\,\hat{\mathbf{b}}$

4. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are any three non-coplanar vectors, then $[\mathbf{a}+\mathbf{b}+\mathbf{c}\;\;\mathbf{a}-\mathbf{c}\;\;\mathbf{a}-\mathbf{b}]$ is equal to
(a) 0 *(b)* $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$
(c) $2[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ *(d)* $-3[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$

5. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the three vectors having magnitudes 1, 1 and 2 respectively such that $\mathbf{a}\times(\mathbf{a}\times\mathbf{c}) + \mathbf{b} = 0$, then the acute angle between $\mathbf{a}$ and $\mathbf{c}$ is
(a) $\frac{\pi}{3}$ *(b)* $\frac{\pi}{4}$ *(c)* $\frac{\pi}{6}$ *(d)* $\frac{\pi}{2}$

6. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar and non-zero vectors, then $(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})\times(\mathbf{b}\times\mathbf{a}) + (\mathbf{c}\times\mathbf{a})\times(\mathbf{c}\times\mathbf{b})$ is equal to
(a) $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2(\mathbf{a}+\mathbf{b}+\mathbf{c})$ *(b)* $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}](\mathbf{a}+\mathbf{b}+\mathbf{c})$
(c) 0 *(d)* None of these

7. The acute angle between any two faces of a regular tetrahedron is
(a) $\cos^{-1}\left(\frac{1}{3}\right)$ *(b)* $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$
(c) $\cos^{-1}\left(\frac{2}{3}\right)$ *(d)* None of these

8. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-null coplanar vectors, then $[\mathbf{a}-2\mathbf{b}+\mathbf{c}\;\;\mathbf{b}-2\mathbf{c}+\mathbf{a}\;\;\mathbf{c}-2\mathbf{a}+\mathbf{b}]$ equals
(a) $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ *(b)* $[\mathbf{a}\,\mathbf{c}\,\mathbf{b}]$ *(c)* 0 *(d)* $12[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$

9. Let G_1, G_2 and G_3 be the centroids of the triangular faces OBC, OCA, OAB of a tetrahedron $OABC$. If V_1 denotes the volume of the tetrahedron $OABC$ and V_2 that of the parallelopiped with OG_1, OG_2 and OG_3 as three concurrent edges, then
(a) $4V_1 = 9V_2$ *(b)* $9V_1 = 4V_2$
(c) $3V_1 = 2V_2$ *(d)* $3V_2 = 2V_1$

10. Let $\mathbf{r}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be four non-zero vectors such that $\mathbf{r}\cdot\mathbf{a} = 0, |\mathbf{r}\times\mathbf{b}| = |\mathbf{r}|\,|\mathbf{b}|, |\mathbf{r}\times\mathbf{c}| = |\mathbf{r}|\,|\mathbf{c}|$, then $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ is equal to
(a) -1 *(b)* 0 *(c)* 1 *(d)* 2

11. Let $\mathbf{V} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{W} = \hat{\mathbf{i}} + 3\hat{\mathbf{k}}$. If $\hat{\mathbf{U}}$ is a unit vector, then the maximum value of the scalar triple product $[\hat{\mathbf{U}}\;\;\mathbf{V}\;\;\mathbf{W}]$, is
(a) -1 *(b)* $\sqrt{10}+\sqrt{6}$ *(c)* $\sqrt{59}$ *(d)* $\sqrt{60}$

12. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are two unit vectors, then the vector $(\hat{\mathbf{a}}+\hat{\mathbf{b}})\times(\hat{\mathbf{a}}\times\hat{\mathbf{b}})$ is parallel to the vector
(a) $\hat{\mathbf{a}}+\hat{\mathbf{b}}$ *(b)* $\hat{\mathbf{a}}-\hat{\mathbf{b}}$ *(c)* $2\hat{\mathbf{a}}+\hat{\mathbf{b}}$ *(d)* $2\hat{\mathbf{a}}-\hat{\mathbf{b}}$

13. If $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are unit vectors. If $\hat{\mathbf{w}}$ is a vector such that $\mathbf{w} + (\mathbf{w}\times\hat{\mathbf{u}}) = \hat{\mathbf{v}}$, then $\hat{\mathbf{u}}\cdot(\hat{\mathbf{v}}\times\mathbf{w})$ will be equal to
(a) $1 - \hat{\mathbf{v}}\cdot\mathbf{w}$ *(b)* $1 - |\mathbf{w}|^2$
(c) $|\mathbf{w}|^2 - (\hat{\mathbf{u}}\cdot\mathbf{w})^2$ *(d)* All of these

14. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other, then a vector $\mathbf{v}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ satisfying the equations $\mathbf{v}\cdot\mathbf{a} = 0$, $\mathbf{v}\cdot\mathbf{b} = 1$ and $[\mathbf{v}\,\mathbf{a}\,\mathbf{b}] = 1$, is
(a) $\frac{\mathbf{b}}{|\mathbf{b}|^2} + \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|^2}$ *(b)* $\frac{\mathbf{b}}{|\mathbf{b}|} + \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|^2}$
(c) $\frac{\mathbf{b}}{|\mathbf{b}|^2} + \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ *(d)* None of these

15. The value of a so that the volume of the parallelopiped formed by $\hat{\mathbf{i}} + a\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{j}} + a\hat{\mathbf{k}}$ and $a\hat{\mathbf{i}} + \hat{\mathbf{k}}$ becomes minimum, is
(a) $\frac{1}{3}$ *(b)* 3 *(c)* $\frac{1}{\sqrt{3}}$ *(d)* $\sqrt{3}$

16. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three non-coplanar vectors, then $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{b}+\mathbf{c})\times(\mathbf{c}+\mathbf{a})$ equals
(a) 0 *(b)* $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ *(c)* $2[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ *(d)* $3[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$

17. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors such that $|\mathbf{b}| = |\mathbf{c}|$, then $\{(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{c})\}\times(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}+\mathbf{c})$ is equal to
(a) 1 *(b)* -1
(c) 0 *(d)* None of these

18. If the volume of the parallelopiped formed by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as three coterminous edges is 27 cu units, then the volume of the parallelopiped having $\alpha = \mathbf{a} + 2\mathbf{b} - \mathbf{c}$, $\beta = \mathbf{a} - \mathbf{b}$ and $\gamma = \mathbf{a} - \mathbf{b} - \mathbf{c}$ as three coterminous edges is
(a) 27 cu units *(b)* 9 cu units
(c) 81 cu units *(d)* None of these

19. $\{\mathbf{a}\cdot(\mathbf{b}\times\hat{\mathbf{i}})\}\,\hat{\mathbf{i}} + \{\mathbf{a}\cdot(\mathbf{b}\times\hat{\mathbf{j}})\}\,\hat{\mathbf{j}} + \{\mathbf{a}\cdot(\mathbf{b}\times\hat{\mathbf{k}})\}\hat{\mathbf{k}}$ is equal to
(a) $2(\mathbf{a}\times\mathbf{b})$ *(b)* $3(\mathbf{a}\times\mathbf{b})$
(c) $\mathbf{a}\times\mathbf{b}$ *(d)* $-(\mathbf{a}\times\mathbf{b})$

20. The unit vector which is orthogonal to the vector $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and is coplanar with vectors $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, is
(a) $\frac{1}{\sqrt{41}}(2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}})$ *(b)* $\frac{1}{\sqrt{13}}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}})$
(c) $\frac{1}{\sqrt{10}}(3\hat{\mathbf{j}} - \hat{\mathbf{k}})$ *(d)* $\frac{1}{\sqrt{34}}(4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$

21. If $\mathbf{a}$ and $\mathbf{b}$ are the vectors in space given by $\mathbf{a}=\frac{\hat{\mathbf{i}}-2\hat{\mathbf{j}}}{\sqrt{5}}$ and $\mathbf{b}=\frac{2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}}{\sqrt{14}}$, then the value of $(2\mathbf{a}+\mathbf{b})\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}-2\mathbf{b})]$ is

(a) 2 (b) 3 (c) 4 (d) 5

22. Two adjacent sides of the parallelogram *ABCD* are given by $\mathbf{AB}=2\hat{\mathbf{i}}+10\hat{\mathbf{j}}+11\hat{\mathbf{k}}$ and $\mathbf{AD}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$. The side *AD* is rotated by an acute angle α in the plane of the parallelogram so that *AD* becomes *AD'*. If *AD'* makes a right angle with the side *AB*, then the cosine of the angle α is given by

(a) $\frac{8}{9}$ (b) $\frac{\sqrt{17}}{9}$ (c) $\frac{1}{9}$ (d) $\frac{4\sqrt{5}}{9}$

23. The vectors which are coplanar with vectors $\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$, and perpendicular to the vector $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ are

(a) $\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\hat{\mathbf{j}}+\hat{\mathbf{k}}$ (b) $-\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $-\hat{\mathbf{i}}-\hat{\mathbf{j}}$
(c) $\hat{\mathbf{i}}-\hat{\mathbf{j}}$ and $\hat{\mathbf{j}}-\hat{\mathbf{k}}$ (d) $-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $-\hat{\mathbf{i}}+\hat{\mathbf{j}}$

24. Let $\mathbf{a}=-\hat{\mathbf{i}}-\hat{\mathbf{k}}$, $\mathbf{b}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ be three given vectors. If $\mathbf{r}$ is a vector such that $\mathbf{r}\times\mathbf{b}=\mathbf{c}\times\mathbf{b}$ and $\mathbf{r}\cdot\mathbf{a}=0$, then the value of $\mathbf{r}\cdot\mathbf{b}$ is

(a) 4 (b) 8 (c) 6 (d) 9

25. If $\mathbf{a}=\frac{1}{\sqrt{10}}(3\hat{\mathbf{i}}+\hat{\mathbf{k}})$, $\mathbf{b}=\frac{1}{7}(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-6\hat{\mathbf{k}})$, then the value of $(2\mathbf{a}-\mathbf{b})\cdot\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}+2\mathbf{b})\}$ is

(a) -5 (b) -3 (c) 5 (d) 3

26. The number of distinct real value of λ for which the vectors $\mathbf{a}=\lambda^3\hat{\mathbf{i}}+\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}-\lambda^3\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{i}}+(2\lambda-\sin\lambda)\hat{\mathbf{j}}-\lambda\hat{\mathbf{k}}$ are coplanar, is

(a) 0 (b) 1 (c) 2 (d) 3

27. The points with position vectors $\alpha\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\beta\hat{\mathbf{k}}$ are coplanar, if

(a) $(1-\alpha)(1+\beta)=0$
(b) $(1-\alpha)(1-\beta)=0$
(c) $(1+\alpha)(1+\beta)=0$
(d) $(1+\alpha)(1+\beta)=0$

28. Let $\mathbf{PR}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and $\mathbf{SQ}=\hat{\mathbf{i}}-3\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ determine diagonals of a parallelogram *PQRS* and $\mathbf{PT}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ be another vector. Then, the volume of the parallelopiped determined by the vectors $\mathbf{PT}$, $\mathbf{PQ}$ and $\mathbf{PS}$ is

(a) 5 (b) 20 (c) 10 (d) 30

29. If $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}]=\lambda[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$, then λ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

30. Let unit vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are coplanar and a unit vector $\hat{\mathbf{d}}$ is perpendicular to them. If $(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\times(\hat{\mathbf{c}}\times\hat{\mathbf{d}})=\frac{1}{6}\hat{\mathbf{i}}-\frac{1}{3}\hat{\mathbf{j}}+\frac{1}{3}\hat{\mathbf{k}}$ and the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is 30°, then $\hat{\mathbf{c}}$ is/are

(a) $\frac{1}{3}(-2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) $\pm\frac{1}{3}(-\hat{\mathbf{i}}-2\hat{\mathbf{j}}+2\hat{\mathbf{k}})$
(c) $\frac{1}{3}(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$ (d) $\pm\frac{1}{3}(-\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}})$

Answers

1. (a)	2. (c)	3. (c)	4. (d)	5. (c)
6. (b)	7. (a)	8. (c)	9. (a)	10. (b)
11. (c)	12. (b)	13. (d)	14. (a)	15. (c)
16. (b)	17. (c)	18. (c)	19. (c)	20. (c)
21. (d)	22. (b)	23. (a)	24. (d)	25. (a)
26. (b)	27. (a)	28. (a)	29. (b)	30. (d)

Continued from Page 40

26. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$, then the value of $p\frac{d^2p}{d\theta^2}$ is equal to

(a) $\frac{a^2b^2}{p^3}$ (b) $\frac{2a^2b^2}{p^2}$ (c) $\frac{3a^2b^3}{p^3}$ (d) $\frac{2a^2b^4}{p^3}$

27. The smallest positive values of x and y satisfying $x-y=\frac{\pi}{4}$, $\cot x+\cot y=2$, are

(a) $x=\frac{5\pi}{12}, y=\frac{\pi}{3}$ (b) $x=\frac{4\pi}{3}, y=\frac{\pi}{6}$
(c) $x=\frac{6\pi}{5}, y=\frac{\pi}{4}$ (d) $x=\frac{5\pi}{12}, y=\frac{\pi}{6}$

28. The value of $\tan^{-1}\sqrt{\left\{\frac{x(x+y+z)}{yz}\right\}}+\tan^{-1}\sqrt{\left\{\frac{y(x+y+z)}{zx}\right\}}+\tan^{-1}\sqrt{\left\{\frac{z(x+y+z)}{xy}\right\}}$ is equal to

(a) π (b) 2π (c) $\frac{1}{2}\pi$ (d) 3π

29. Let $n(U)=700$, $n(A)=200$, $n(B)=300$ and $n(A\cap B)=100$. Then, $n(A^C\cap B^C)$ is equal to

(a) 400 (b) 600
(c) 300 (d) 200

30. The value of $\lim_{x\to 0}\frac{1+\sin x-\cos x+\log_e(1-x)}{x^3}$, is

(a) -1 (b) 1/2 (c) $-1/2$ (d) 1

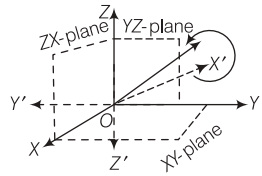
Answers

1. (c)	2. (b)	3. (d)	4. (a)	5. (d)
6. (a)	7. (b)	8. (b)	9. (a)	10. (a)
11. (a)	12. (c)	13. (b)	14. (b)	15. (d)
16. (a)	17. (b)	18. (b)	19. (c)	20. (a)
21. (d)	22. (c)	23. (a)	24. (a)	25. (c)
26. (a)	27. (d)	28. (a)	29. (c)	30. (c)

3D GEOMETRY

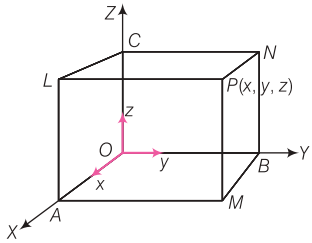
Coordinate Axes and Coordinate Planes in 3D

Three mutually perpendicular lines intersecting at O in space, divide the space into eight parts which are known as octants. The perpendicular lines are known as coordinate axes and the planes XY, YZ and ZX are known as coordinate planes.



Coordinate of a Point in Space

Let P be a point in space, then the position of P is given by triad (x, y, z), where x, y and z are perpendicular distances of P from YZ-plane, ZX-plane and XY-plane, respectively. If we assume $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ unit vectors along OX, OY and OZ respectively, then position vector of point P is $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.



Note The coordinates of origin O are (0, 0, 0). The coordinates of any point on X-axis are $(x, 0, 0)$, on Y-axis are $(0, y, 0)$ and on Z-axis are $(0, 0, z)$. The coordinates of any point in YZ-plane are $(0, y, z)$, in XY-plane are $(x, y, 0)$ and in ZX-plane are $(x, 0, z)$.

Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. Then, the distance between these points is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Section Formulae

(i) The coordinates of any point, which divide the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$ internally, are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$.

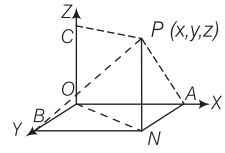
(ii) The coordinates of any point, which divide the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$ externally, are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$.

REMEMBER

- The coordinates of mid-point of line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.
- The coordinates of the centroid of a triangle formed with vertices $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$, are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.
- The coordinates of the centroid of a tetrahedron formed with vertices $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ and $S(x_4, y_4, z_4)$ are $\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$.

SPEED BOOSTER

- If the position vector $\mathbf{r}$ of point P is given by $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, then coordinates of P are (x, y, z).
- The distance of any point $P(x, y, z)$ having position vector $\mathbf{OP} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ from the origin is $\sqrt{x^2 + y^2 + z^2}$ which is equivalent to the modulus of the vector $\mathbf{OP}$.
- Let $P(x, y, z)$ be any point in the space and PA, PB, PC be the perpendicular drawn from P to OX, OY, OZ-axes, respectively.



Then, $PA = \sqrt{(y^2 + z^2)}$, $PB = \sqrt{(z^2 + x^2)}$

and $PC = \sqrt{(x^2 + y^2)}$

- The coordinates of any point lying on the line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$, which divide PQ in the ratio $k : 1$. This is called general point on the line PQ.

Direction Cosines of a Line

Let α, β and γ be the angles which a directed line makes with the positive direction of X, Y and Z-axes respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines of the given line and generally they are denoted by l, m and n respectively, i.e. $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

Note (i) The direction cosines of X-axis are (1, 0, 0), Y-axis are (0, 1, 0) and Z-axis are (0, 0, 1).

(ii) The unit vector along the line can be taken as $l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$, where l, m, n are direction cosines of line.

(iii) $l^2 + m^2 + n^2 = 1$, where l, m, n are direction cosines of any line.

Direction Ratios of a Line

If a, b and c are three numbers proportional to direction cosines l, m and n of a line, then a, b and c are called its direction ratios. They are also called direction numbers or direction components. Now, by definition, we have

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=k \quad \text{[say]}$$

$$\Rightarrow \quad l=ak, m=bk \text{ and } n=ck \Rightarrow l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}},$$

$$m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}} \text{ and } n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

where, the sign should be taken all positive or all negative.

Note Direction ratios are not unique, whereas direction cosines are unique.

REMEMBER

- The direction ratios of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are x_2-x_1, y_2-y_1 and z_2-z_1.
- If $\mathbf{r}=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$ having direction cosines l, m and n, then
$$l=\frac{a}{|\mathbf{r}|};\ m=\frac{b}{|\mathbf{r}|} \text{ and } n=\frac{c}{|\mathbf{r}|}$$
- If direction ratios of $\mathbf{r}$ are a, b and c, then
$$\mathbf{r}=\frac{|\mathbf{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}})$$

Angle between Two Lines

(i) If θ is the angle between two lines having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, then
$$\cos\theta=l_1l_2+m_1m_2+n_1n_2$$
(a) Lines are parallel, if $\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$.

(b) Lines are perpendicular, if $l_1l_2+m_1m_2+n_1n_2=0$.

(ii) If θ is the angle between two lines whose direction ratios are proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then the angle θ is given by
$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$
(a) Lines are parallel, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

(b) Lines are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$.

SPEED BOOSTER

- Any line equally inclined to all the three axes have direction cosines $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$.
- The angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.
- The angle between a diagonal of a cube and the diagonal of a face of the cube is $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$.
- If the edges of a rectangular parallelopiped are a, b, c, then the angles between the two diagonals are $\cos^{-1}\left(\frac{\pm a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right)$.

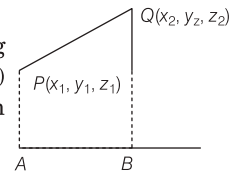
REMEMBER

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the two lines inclined to each other at angle θ, then the direction cosines of

- acute angle bisector are
$$\frac{l_1+l_2}{2\cos\left(\frac{\theta}{2}\right)}, \frac{m_1+m_2}{2\cos\left(\frac{\theta}{2}\right)} \text{ and } \frac{n_1+n_2}{2\cos\left(\frac{\theta}{2}\right)}$$
- obtuse angle bisector are
$$\frac{l_1-l_2}{2\sin\left(\frac{\theta}{2}\right)}, \frac{m_1-m_2}{2\sin\left(\frac{\theta}{2}\right)} \text{ and } \frac{n_1-n_2}{2\sin\left(\frac{\theta}{2}\right)}$$

Projection of a Line Segment on a Given Line

Projection of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line (AB) having direction cosines l, m and n, is $(x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)n$.



Note Here, PQ and AB need not be coplanar.

REMEMBER

- Projection of PQ on X, Y and Z-axes are x_2-x_1, y_2-y_1 and z_2-z_1, respectively.
- Projection of vector $\mathbf{r}$ on the coordinate axes are $l|\mathbf{r}|$; $m|\mathbf{r}|$ and $n|\mathbf{r}|$.

Area of a Triangle

If the vertices of a triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then

$$\text{Area of } \Delta ABC=\sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}$$

where, $$\Delta_x=\frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1\\ y_2 & z_2 & 1\\ y_3 & z_3 & 1\end{vmatrix}$$

$$\Delta_y=\frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1\\ x_2 & z_2 & 1\\ x_3 & z_3 & 1\end{vmatrix} \text{ and } \Delta_z=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

Vector Form

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of ΔABC, then

$$\text{Area of triangle}=\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|=\frac{1}{2}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix}$$

Note Three points are collinear iff area of triangle formed by these points is zero.

MASTER STROKES

1. The cosine of the angle of the triangle with vertices $A(1, -1, 2)$, $B(6, 11, 2)$ and $C(1, 2, 6)$, is

(a) $\frac{63}{65}$ (b) $\frac{36}{65}$ (c) $\frac{16}{65}$ (d) $\frac{13}{64}$

2. In ΔABC, if the mid-point of the sides AB, BC and CA are $(l, 0, 0)$, $(0, m, 0)$ and $(0, 0, n)$ respectively, then $\frac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$ equals

(a) 2 (b) 4 (c) 8 (d) 16

3. If a line makes angles α, β and γ with the coordinate axes respectively, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ equals

(a) 2 (b) -1 (c) 1 (d) -2

4. If a line makes angles $\frac{\pi}{3}$ and $\frac{\pi}{4}$ with X and Y-axes respectively, then the angle made by the line with Z-axis is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{12}$

5. If OA is equally inclined to OX, OY, OZ and A is $\sqrt{3}$ units from the origin, then the coordinates of A are

(a) $(3, 3, 3)$ (b) $(-1, 1, -1)$
(c) $(-1, 1, 1)$ (d) $(1, 1, 1)$

6. If the direction cosines of two lines are connected by the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$, then the angle between the lines is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

7. For every point $P(x, y, z)$ on the XY-plane,

(a) $x = 0$ (b) $y = 0$
(c) $z = 0$ (d) None of these

8. For every point $P(x, y, z)$ on the X-axis (except the origin),

(a) $x = 0, y = 0, z \neq 0$ (b) $x = 0, z = 0, y \neq 0$
(c) $y = 0, z = 0, x \neq 0$ (d) None of these

9. A rectangular parallelopiped is formed by planes drawn through the points (5, 7, 9) and (2, 3, 7), parallel to the coordinate planes. The length of an edge of this rectangular parallelopiped is

(a) 2 (b) 3 (c) 4 (d) All of these

10. A parallelopiped is formed by planes drawn through the points (2, 3, 5) and (5, 9, 7), parallel to the coordinate planes. The length of a diagonal of the parallelopiped is

(a) 7 (b) $\sqrt{38}$
(c) $\sqrt{155}$ (d) None of these

11. The XY-plane divides the line joining the points $(-1, 3, 4)$ and $(2, -5, 6)$

(a) internally in the ratio 2 : 3 (b) externally in the ratio 2 : 3
(c) internally in the ratio 3 : 2 (d) externally in the ratio 3 : 2

12. The points $A(5, -1, 1)$, $B(7, -4, 7)$, $C(1, -6, 10)$ and $D(-1, -3, 4)$ are the vertices of a

(a) trapezium (b) rectangle
(c) rhombus (d) square

13. A straight line which makes an angle of 60° with each of Y and Z-axes, inclines with X-axis at an angle of

(a) 30° (b) 60° (c) 75° (d) 45°

14. If the direction cosines of a line are $\frac{1}{c}, \frac{1}{c}$ and $\frac{1}{c}$, then

(a) $0 < c < 1$ (b) $c > 2$ (c) $c > 0$ (d) $c = \pm\sqrt{3}$

15. The acute angle between the lines whose direction ratios are given by $l + m - n = 0$ and $l^2 + m^2 - n^2 = 0$, is

(a) 0 (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

16. If the direction ratios of two lines are given by $a + b + c = 0$ and $2ab + 2ac - bc = 0$, then the angle between the lines is

(a) π (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

17. The angle between a line with direction ratios proportional to 2, 2, 1 and a line joining (3, 1, 4) to (7, 2, 12), is

(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(-\frac{2}{3}\right)$
(c) $\tan^{-1}\left(\frac{2}{3}\right)$ (d) None of these

18. The projection of the line joining the points (3, 4, 5) and (4, 6, 3) on the line joining the points $(-1, 2, 4)$ and (1, 0, 5), is

(a) $\frac{4}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

19. If the projection of a line segment on the coordinate axes are 2, 3 and 6, then the length of the line segment is

(a) 7 (b) 5 (c) 1 (d) 11

20. A line makes the same angle θ, with each of the X and Z-axes. If the angle β, which it makes with Y-axis, is such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals

(a) $\frac{2}{5}$ (b) $\frac{1}{5}$ (c) $\frac{3}{5}$ (d) $\frac{2}{3}$

21. A line AB in three dimensional space makes angles 45° and 120° with the positive X-axis and the positive Y-axis, respectively. If AB makes an acute angle θ with the positive Z-axis, then θ equals

(a) 60° (b) 75° (c) 30° (d) 45°

22. The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$, is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

Answers

1. (b)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (d)
6. (b)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (a)
11. (b)	**12.** (c)	**13.** (d)	**14.** (d)	**15.** (d)
16. (b)	**17.** (a)	**18.** (a)	**19.** (a)	**20.** (c)
21. (a)	**22.** (c)			

STRAIGHT LINE AND PLANE

Equations of Straight Line in Different Forms

Vector Form

(i) If a straight line passing through a point with position vector $\mathbf{a} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$ and parallel to a vector $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then its vector equation is $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{b}$, where λ is a scalar.

(ii) If a straight line passing through two points having position vectors $\mathbf{a} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$ and $\mathbf{b} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$, then its vector equation is $\mathbf{r} = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})$, where λ is a scalar.

Note The equation of straight line passing through origin and parallel to $\mathbf{b}$ is $\mathbf{r} = \lambda\,\mathbf{b}$.

Cartesian Form

If a line passes through a point $A(x_1, y_1, z_1)$ and having direction ratios a, b, c, then its cartesian equation is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

Note (i) If l, m, n are direction cosines, then equation of straight line is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$, which is known as symmetrical form of line.

(ii) If a line passes through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then its cartesian equation is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

REMEMBER

- The parametric equation of a line passing through $A(x_1, y_1, z_1)$ and having direction ratios a, b, c, is $x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c$, where λ being a parameter.
- The coordinates of any point on the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ is $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where λ being a parameter.

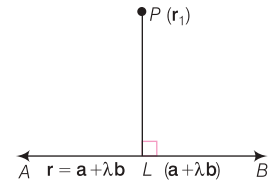
SPEED BOOSTER

- Equation of X-axis is $\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0}$ or $y = 0, z = 0$
- Equation of Y-axis is $\frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0}$ or $x = 0, z = 0$
- Equation of Z-axis is $\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1}$ or $x = 0, y = 0$
- The coordinates of point on the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ which is at a distance of 'r' units from (x_1, y_1, z_1), are $(x_1 + ar, y_1 + br, z_1 + cr)$.

Perpendicular Distance of a Point from a Line

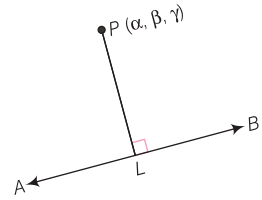
Vector Form

Length of the perpendicular from a point $P(\mathbf{r}_1)$ to the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ is given by $\frac{|(\mathbf{a} - \mathbf{r}_1) \times \mathbf{b}|}{|\mathbf{b}|}$ and foot of perpendicular is given by $\mathbf{a} - \left(\frac{(\mathbf{a} - \mathbf{r}_1) \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$



Cartesian Form

Let the equation of the line be $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ and $P(\alpha, \beta, \gamma)$ be the given point.



To find the length of perpendicular PL, use the following steps :

(i) Let the coordinates of point L be $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$.

(ii) Find DR's of PL and use the condition of perpendicularity to find λ.

(iii) Put the value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, to get the foot of perpendicular.

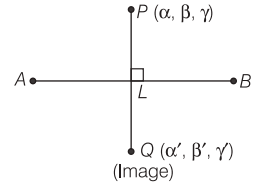
(iv) Find PL using distance formula.

Note The distance of the point (x_2, y_2, z_2) from the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$, where l, m and n are direction cosines of the line, is

$$\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\} - \{l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)\}^2}$$

SPEED BOOSTER

Image of a point in the given line



Since, L (foot of perpendicular) is the mid-point of P and Q (image of a point P in the line), we can get Q, if L is found out.

Skew-lines

Two straight lines which are not parallel and which do not intersect, are known as skew-lines.

Note Two skew-lines are never coplanar.

Line of Shortest Distance

If l_1 and l_2 are two skew-lines, then the straight line which is perpendicular to each of these two lines, is called the line of shortest distance.

Note There is one and only one line perpendicular to each of the lines l_1 and l_2.

Shortest Distance between Two Skew-lines

Vector Form

The shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ is

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1)\cdot(\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

Condition for Lines to Intersect

The two lines are intersecting, if

$$\frac{|(\mathbf{b}_1 \times \mathbf{b}_2)\cdot(\mathbf{a}_2 - \mathbf{a}_1)|}{|\mathbf{b}_1 \times \mathbf{b}_2|} = 0$$

$$\Rightarrow \quad (\mathbf{b}_1 \times \mathbf{b}_2)\cdot(\mathbf{a}_2 - \mathbf{a}_1) = 0$$

Cartesian Form

The shortest distance between the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and

$$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

is given by

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - c_1b_2)^2 + (c_1a_2 - a_1c_2)^2 + (a_1b_2 - b_1a_2)^2}}$$

Condition for Lines to Intersect

The lines are intersecting, if shortest distance = 0, i.e.

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

REMEMBER

The distance or shortest distance between two parallel lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu\,\mathbf{b}$ is given by $\frac{|\mathbf{b} \times (\mathbf{a}_2 - \mathbf{a}_1)|}{|\mathbf{b}|}$.

Plane

A plane is a surface such that the line segment joining any two points on it, lies completely on it.

Equation of Plane in Different Forms

The general equation of a plane is $ax + by + cz + d = 0$, where a, b, c are DR's of the normal to the plane and $a^2 + b^2 + c^2 \neq 0$.

Equation of Plane in Normal Form

Vector Form

The vector equation of a plane normal to unit vector $\hat{\mathbf{n}}$ at a distance d from the origin is $\mathbf{r}\cdot\hat{\mathbf{n}} = d$.

Cartesian Form

The equation of plane in normal form is $lx + my + nz = p$, where l, m and n are direction cosines of a line normal to plane and p is perpendicular distance of the plane from origin.

Intercept Form

The equation of a plane whose intercepts are a, b and c on X, Y and Z-axes respectively, is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Equation of Plane Passing through a Given Point

Vector Form

The vector equation of a plane passing through a given point having position vector $\mathbf{a}$, is $(\mathbf{r} - \mathbf{a})\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is a vector normal to the plane.

Cartesian Form

The equation of a plane passing through a given point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b and c are direction ratios of a line perpendicular to plane.

Equation of Plane Passing through Three Non-collinear Points

Vector Form

The equation of a plane passing through three non-collinear points having position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ is $[\mathbf{r} - \mathbf{r}_1 \;\; \mathbf{r} - \mathbf{r}_2 \;\; \mathbf{r} - \mathbf{r}_3] = 0$.

Cartesian Form

The equation of plane passing through three non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{vmatrix} = 0$$

Equation of Plane through Two Given Points and Parallel to a Given Vector (or Line)

Vector Form

The equation of a plane passing through two given points having position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and parallel to a given vector $\mathbf{m}$, is $(\mathbf{r} - \mathbf{r}_1)\cdot[(\mathbf{r} - \mathbf{r}_2) \times \mathbf{m}] = 0$.

Cartesian Form

The equation of a plane passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) and parallel to a line having direction ratios a, b, c, is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ a & b & c \end{vmatrix} = 0.$

Equation of Plane Passing through a Given Point and Parallel to Two Given Vectors (or Lines)

Vector Form

The equation of a plane passing through a point having position vector $\mathbf{a}$ and parallel to two given vectors $\mathbf{b}$ and $\mathbf{c}$, is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where λ and μ are scalars.
or $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c}) = 0$ or $\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

Cartesian Form

The equation of a plane passing through a point (x_1, y_1, z_1) and parallel to two lines having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2, is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

Planes Parallel to a Given Plane

Vector Form

The equation of a plane parallel to the plane $\mathbf{r} \cdot \hat{\mathbf{n}} = d_1$ is $\mathbf{r} \cdot \hat{\mathbf{n}} = d_2$, where d_2 is a constant to be determined by the given condition.

Cartesian Form

The equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is a constant to be determined by the given condition.

REMEMBER

- Equation of XY-plane is $z = 0$, equation of plane parallel to XY-plane is $z = d$.
- Equation of YZ-plane is $x = 0$, equation of plane parallel to YZ-plane is $x = d$.
- Equation of ZX-plane is $y = 0$, equation of plane parallel to ZX-plane is $y = d$.
- Four points namely $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ will be coplanar, if one point lies on the plane passing through other three points.

Planes Passing through the Intersection of Two Planes

Vector Form

The equation of a plane passing through the intersection of the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ is $(\mathbf{r} \cdot \mathbf{n}_1 - d_1) + k(\mathbf{r} \cdot \mathbf{n}_2 - d_2) = 0$ or $\mathbf{r} \cdot (\mathbf{n}_1 + k\mathbf{n}_2) = d_1 + kd_2$, where k is any arbitrary constant.

Cartesian Form

The equation of a plane passing through the intersection of planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$
where, k is any arbitrary constant.

REMEMBER

If $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are two non-parallel planes, then these two equations taken together represent a line, which is known as unsymmetrical form of a line.

Distance of a Point from a Plane

Vector Form

The length of the perpendicular from a point having position vector $\mathbf{a}$ to the plane $\mathbf{r} \cdot \mathbf{n} = d$, is given by $p = \frac{|\mathbf{a} \cdot \mathbf{n} - d|}{|\mathbf{n}|}$.

Cartesian Form

The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Distance between Two Parallel Planes

Vector Form

The distance between two parallel planes $\mathbf{r} \cdot \mathbf{n} = d_1$ and $\mathbf{r} \cdot \mathbf{n} = d_2$ is given by $p = \frac{|d_1 - d_2|}{|\mathbf{n}|}$.

Cartesian Form

The distance between two parallel planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$ is given by $p = \frac{|d_1 - d_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$.

Position of Two Points with Respect to a Plane

The two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same side or the opposite sides of the plane $ax + by + cz + d = 0$, according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ have the same sign or the opposite signs.

Angle between Two Planes

Vector Form

Let θ be the angle between the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$, then $\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$.

Cartesian Form

The angle between the planes

$$a_1x + b_1y + c_1z + d_1 = 0$$

and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

REMEMBER

Two planes are perpendicular, if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel, if $\frac{\mathbf{n}_1}{\mathbf{n}_2} = \lambda$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Planes Bisecting the Angles between Two Planes

Vector Form

The equation of the planes bisecting the angles between the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ are

$$\frac{|\mathbf{r} \cdot \mathbf{n}_1 - d_1|}{|\mathbf{n}_1|} = \frac{|\mathbf{r} \cdot \mathbf{n}_2 - d_2|}{|\mathbf{n}_2|} \text{ or } \mathbf{r} \cdot (\hat{\mathbf{n}}_1 \pm \hat{\mathbf{n}}_2) = \frac{d_1}{|\mathbf{n}_1|} \pm \frac{d_2}{|\mathbf{n}_2|}$$

Cartesian Form

The equations of the planes bisecting the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

REMEMBER

If $d_1, d_2 > 0$, then

- The equation of the bisector of the angle between the two planes containing the origin is
$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$
- If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then the origin lies in the obtuse angle between the two planes and the equation of the bisector of the acute angle is
$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$
- If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then the origin lies in the acute angle between the two planes and the equation of the bisector of the obtuse angle between the two planes is
$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Angle between a Line and a Plane

The angle between a line and a plane is the angle between the line and normal to the plane.

Vector Form

If θ is the angle between the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane $\mathbf{r} \cdot \mathbf{n} = d$, then $\sin\theta = \frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$.

Cartesian Form

If θ is the angle between the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and the plane $a_2x + b_2y + c_2z + d = 0$, then

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note (i) If the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is parallel to the plane $a_2x + b_2y + c_2z + d = 0$, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(ii) If line is perpendicular to the plane, then we have $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1}$

Condition for a Line to Lie in a Plane

Vector Form

The line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ lies in the plane $\mathbf{r} \cdot \mathbf{n} = d$ iff $\mathbf{b} \cdot \mathbf{n} = 0$ and $\mathbf{a} \cdot \mathbf{n} = d$.

Cartesian Form

The line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ lies in the plane $ax + by + cz + d = 0$ iff

(i) $ax_1 + by_1 + cz_1 + d = 0$ (ii) $al + bm + cn = 0$

SPEED BOOSTER

The equation of a plane containing the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where $al + bm + cn = 0$

Condition for Two Non-parallel Lines to be Intersecting (Coplanar) and the Equation of Plane Containing Them

Vector Form

The lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ are intersecting (coplanar), iff $[\mathbf{a}_1\ \mathbf{b}_1\ \mathbf{b}_2] = [\mathbf{a}_2\ \mathbf{b}_1\ \mathbf{b}_2]$ and the equation of the plane containing the two lines is $[\mathbf{r}\ \mathbf{b}_1\ \mathbf{b}_2] = [\mathbf{a}_1\ \mathbf{b}_1\ \mathbf{b}_2]$ or $[\mathbf{r}\ \mathbf{b}_1\ \mathbf{b}_2] = [\mathbf{a}_2\ \mathbf{b}_1\ \mathbf{b}_2]$

Cartesian Form

The lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

are intersecting (coplanar) iff $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

and the equation of the plane containing the two lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

SPEED BOOSTER

- The image or reflection (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$
- The foot of perpendicular (x, y, z) from a point (x_1, y_1, z_1) on the plane $ax + by + cz + d = 0$ is given by
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

MASTER STROKES

1. If the line passing through the points $(5,1,a)$ and $(3,b,1)$ crosses the YZ-plane at the point $\left(0,\frac{17}{2},-\frac{13}{2}\right)$, then

(a) $a=6, b=4$ (b) $a=8, b=2$
(c) $a=2, b=8$ (d) $a=4, b=6$

2. If the straight lines $\frac{x-1}{k}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x-2}{3}=\frac{y-3}{k}=\frac{z-1}{2}$ intersect at a point, then k is equal to

(a) 2 (b) -2 (c) -5 (d) 5

3. Let the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lie in the plane $x+3y-\alpha z+\beta=0$. Then, (α,β) equals

(a) $(6,-17)$ (b) $(-6,7)$
(c) $(5,-15)$ (d) $(-5,5)$

4. Consider the lines $L_1: \frac{x+1}{3}=\frac{y+2}{1}=\frac{z+1}{2}$ and $L_2: \frac{x-2}{1}=\frac{y+2}{2}=\frac{z-3}{3}$. The unit vector perpendicular to both L_1 and L_2 is

(a) $\frac{1}{\sqrt{99}}(-\hat{\mathbf{i}}+7\hat{\mathbf{j}}+7\hat{\mathbf{k}})$ (b) $\frac{1}{5\sqrt{5}}(-\hat{\mathbf{i}}-7\hat{\mathbf{j}}+5\hat{\mathbf{k}})$
(c) $\frac{1}{5\sqrt{3}}(-\hat{\mathbf{i}}+7\hat{\mathbf{j}}+5\hat{\mathbf{k}})$ (d) $\frac{1}{\sqrt{99}}(7\hat{\mathbf{i}}-7\hat{\mathbf{j}}-\hat{\mathbf{k}})$

5. Let $P(3,2,6)$ be a point in space and θ be a point on the line $\mathbf{r}=(\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})+\mu(-3\hat{\mathbf{i}}+\hat{\mathbf{j}}+5\hat{\mathbf{k}})$. Then, the value of μ for which the vector $\mathbf{PQ}$ is parallel to the plane $x-4y+3z=1$, is

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

6. A plane passes through $(1,-2,1)$ and is perpendicular to two planes $2x-2y+z=0$ and $x-y+2z=4$. The distance of the plane from the point $(1,2,2)$ is

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) $2\sqrt{2}$

7. Let $\mathbf{A}$ be a vector parallel to line of intersection of planes P_1 and P_2 through the origin. If P_1 is parallel to the vectors $2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $4\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and P_2 is parallel to $\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}+3\hat{\mathbf{j}}$, then the angle between $\mathbf{A}$ and $2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ is

(a) $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ (c) $\frac{\pi}{6}$ or $\frac{\pi}{3}$ (d) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$

8. The point (α,β,γ) lies on the plane $x+y+z=2$. Let $\mathbf{a}=\alpha\hat{\mathbf{i}}+\beta\hat{\mathbf{j}}+\gamma\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{a})=0$, then γ equals

(a) 0 (b) 1 (c) 2 (d) 3

9. A variable plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ at a unit distance from the origin cuts the coordinate axes at A, B and C. If centroid (x,y,z) of ΔABC satisfies the equation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, then the value of k is

(a) 9 (b) 3 (c) 1/9 (d) 1/3

10. If the distance of the point $P(1,-2,1)$ from the plane $x+2y-2z=\alpha$, where $\alpha>0$, is 5, then the foot of the perpendicular from P to the plane is

(a) $\left(\frac{8}{3},\frac{4}{3},\frac{-7}{3}\right)$ (b) $\left(\frac{4}{3},\frac{-4}{3},\frac{1}{3}\right)$
(c) $\left(\frac{1}{3},\frac{2}{3},\frac{10}{3}\right)$ (d) $\left(\frac{2}{3},-\frac{1}{3},\frac{5}{2}\right)$

11. The equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$, is

(a) $x+2y-2z=0$ (b) $3x+2y-2z=0$
(c) $x-2y+z=0$ (d) $5x+2y-4z=0$

12. If the distance between the plane $Ax-2y+z=d$ and the plane containing the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is equal to

(a) 3 (b) 4 (c) 6 (d) 1

13. A line from the origin meets the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2}=\frac{y+3}{-1}=\frac{z-1}{1}$ at P and Q respectively. If length $PQ=d$, then d^2 is equal to

(a) 3 (b) 4
(c) 5 (d) 6

14. The distance of the point $(1,-5,9)$ from the plane $x-y+z=5$ measured along the straight line $x=y=z$ is

(a) $5\sqrt{3}$ (b) $3\sqrt{10}$ (c) $3\sqrt{5}$ (d) $10\sqrt{3}$

15. The length of the perpendicular drawn from the point $(3,-1,11)$ to the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$, is

(a) $\sqrt{33}$ (b) $\sqrt{53}$ (c) $\sqrt{66}$ (d) $\sqrt{29}$

16. The point P is the intersection of the straight line joining the points $Q(2,3,5)$ and $R(1,-1,4)$ with the plane $5x-4y-z=1$. If S is the foot of the perpendicular drawn from the point $T(2,1,4)$ to QR, then the length of the line segment PS is

(a) $1/\sqrt{2}$ (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$

17. The equation of a plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$ and at a distance $2/\sqrt{3}$ from the point $(3,1,-1)$, is

(a) $5x-11y+z=17$ (b) $\sqrt{2}x+y=3\sqrt{2}-1$

(c) $x+y+z=\sqrt{3}$ (d) $x-\sqrt{2}y=1-\sqrt{2}$

18. If the straight lines $\frac{x-1}{2}=\frac{y+1}{k}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$ are coplanar, then the planes containing these two lines are

(a) $y\pm 2z=-1$ (b) $y\pm z=-1$

(c) $y\pm 2z=1$ (d) $y\pm z=1$

19. If the three planes $x=5$, $2x-5ay+3z-2=0$ and $3bx+y-3z=0$ contain a common line, then (a,b) is equal to

(a) $\left(-\frac{1}{5},\frac{8}{15}\right)$ (b) $\left(\frac{1}{5},-\frac{8}{15}\right)$

(c) $\left(\frac{-8}{15},\frac{1}{5}\right)$ (d) $\left(\frac{8}{15},-\frac{1}{5}\right)$

20. If perpendicular are drawn from the points on the line $\frac{x+2}{2}=\frac{y+1}{-1}=\frac{z}{3}$ to the plane $x+y+z=3$, then the foot of the perpendicular lie on the line

(a) $\frac{x}{5}=\frac{y-1}{8}=\frac{z-2}{-13}$ (b) $\frac{x}{2}=\frac{y-1}{3}=\frac{z-2}{-5}$

(c) $\frac{x}{4}=\frac{y-1}{3}=\frac{z-2}{-7}$ (d) $\frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$

21. If a line l passing through the origin is perpendicular to the lines

$l_1:\mathbf{r}=(3+t)\,\hat{\mathbf{i}}+(-1+2t)\hat{\mathbf{j}}+(4+2t)\hat{\mathbf{k}}$

$l_2:\mathbf{r}=(3+2t)\,\hat{\mathbf{i}}+(3+2t)\hat{\mathbf{j}}+(2+t)\hat{\mathbf{k}}$

Then, the coordinates of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1, are

(a) $\left(\frac{7}{3},\frac{7}{3},\frac{5}{3}\right)$ and $(-1,-1,0)$ (b) $(-1,-1,0)$ and $(1,1,1)$

(c) $\left(\frac{7}{9},\frac{7}{9},\frac{8}{9}\right)$ and $\left(\frac{7}{3},\frac{7}{3},\frac{5}{3}\right)$ (d) $(1,1,1)$ and $\left(\frac{7}{9},\frac{7}{9},\frac{8}{9}\right)$

22. Consider the set of eight vectors $V=\{a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}:a,b,c\in(-1,1)\}$.

If three non-coplanar vectors can be chosen from V in 2^n ways, then n equals

(a) 3 (b) 5

(c) 6 (d) 4

23. If the lines $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-\lambda}{3}$ and $\frac{x}{1}=\frac{y+2}{2}=\frac{z}{4}$ intersect each other, then λ lies in the interval

(a) (9, 11) (b) (–5, –3)

(c) (13, 15) (d) (11, 13)

24. The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3=0$ is the line

(a) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ (b) $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$

(c) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (d) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$

25. From a point $P(\lambda,\lambda,\lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y=x$, $z=1$ and $y=-x$, $z=-1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is/are

(a) $\sqrt{2}$ (b) 1 (c) -1 (d) $-\sqrt{2}$

26. A variable plane is at a constant distance p from the origin O and meets the set of rectangular axes $OX_i (i=1,2,3)$ at points $A_i (i=1,2,3)$, respectively. If planes are drawn through A_1, A_2, A_3 which are parallel to the coordinate planes, then the locus of their point of intersection is

(a) $x^2+y^2+z^2=p^2$ (b) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{p}$

(c) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$ (d) $\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}=\frac{1}{p^3}$

27. The equation of the plane containing the lines $2x-5y+z=3$; $x+y+4z=5$ and parallel to the plane $x+3y+6z=1$, is

(a) $x+3y+6z=7$ (b) $2x+6y+12z=-13$

(c) $2x+6y+12z=13$ (d) $x+3y+6z=-7$

28. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=16$, is

(a) $3\sqrt{21}$ (b) 13 (c) $2\sqrt{14}$ (d) 8

29. The distance between the lines $x=2+t$, $y=1+t$, $z=-\frac{1}{2}-\frac{t}{2}$ and the plane $\mathbf{r}\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})=10$, is

(a) $\frac{1}{6}$ (b) $\frac{1}{\sqrt{41}}$

(c) $\frac{1}{7}$ (d) $\frac{9}{\sqrt{41}}$

30. The shortest distance between the lines $2x+y+z-1=0$, $3x+y+2z-2=0$ and $x=y=z$, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) $\frac{3}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

Answers

1. (a)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (a)
6. (d)	**7.** (a)	**8.** (c)	**9.** (a)	**10.** (a)
11. (c)	**12.** (c)	**13.** (d)	**14.** (d)	**15.** (b)
16. (a)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (d)
21. (d)	**22.** (b)	**23.** (d)	**24.** (c)	**25.** (c)
26. (c)	**27.** (a)	**28.** (b)	**29.** (d)	**30.** (a)

PROBABILITY

Probability is defined as a quantitative measure of uncertainty. It is an approximate tool for many diverse field.

Terminology used in Probability

Sample Space

The set of all possible outcomes of a trial (random experiment) is called its sample space.

In general, it is denoted by S and outcomes of the trial are known as sample points.

Event

The subset of sample space is called an event.

Note (i) Sample space S plays the same role as the universal set for all problems related to the particular experiment.
(ii) ϕ is also a subset of S which is called an impossible event.
(iii) S is also a subset of S which is called a sure event or a certain event.

Types of Event

Simple Event

An event containing only a single sample point is called simple event or elementary event.

Compound Event

Event obtained by combining two or more elementary events is called compound event.

Equally Likely Events

Outcomes are said to be equally likely, when we have no reason to believe that one is more likely to occur than the other.

Mutually Exclusive Events

Events are said to be mutually exclusive or disjoint, if occurrence of any one of them prevents the occurrence of all the others.

Independent Events

Events are said to be independent, if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of others.

Dependent Events

Two or more events are said to be dependent, if the happening of one event affects (partially or totally) the happening of other events.

Exhaustive Events

If $E_1, E_2, E_3, \ldots, E_n$ are n events of a sample S and

$$S = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = \bigcup_{i=1}^{n} E_i, \text{ then}$$

$E_1, E_2, E_3, \ldots, E_n$ are called exhaustive events.

In other words, events $E_1, E_2, E_3, \ldots, E_n$ are said to be exhaustive, if atleast one of them necessarily occurs whenever the experiment is performed.

Mutually Exclusive and Exhaustive Events

A set of events is said to be mutually exclusive and exhaustive, if following conditions are satisfied.

(i) $E_1 \cup E_2 \cup E_3 \cup E_4 \cup \ldots \cup E_n = S$
(ii) $E_1 \cap E_2 \cap E_3 \cap E_4 \cap \ldots \cap E_n = \phi$

Note The collection of elementary events associated with a random experiment always form a system of mutually exclusive and exhaustive system of experiments. In this system

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n) = 1$$

Axiomatic Approach of Probability

In this approach, some axioms or rules are depicted to assign probabilities.

Let $S = \{w_1, w_2, w_3, \ldots, w_n\}$ be the sample space associated with a random experiment. Then, a function P which assigns every event $E \subset S$ to a unique non-negative real number $P(E)$, is called probability function, if the following axioms hold

(i) $0 \le P(w_i) \le 1$ for all $w_i \in S$
(ii) $P(S) = 1$, i.e. $P(w_1) + P(w_2) + \ldots + P(w_n) = 1$
(iii) For any event $E \subset S$, $P(E) = \sum_{w_i \in A} P(w_i)$, the number $P(w_i)$ is called the probability of elementary event w_i.

Classical Definition of Probability

Let S be the sample space, then the probability of occurrence of an event E is denoted by $P(E)$ and is defined as

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of elements in } E}{\text{Number of elements in } S} = \frac{\text{Number of cases favourable to } E}{\text{Total number of cases}}$$

REMEMBER

- $0 \le P(E) \le 1$
- $P(E) + P(\bar{E}) = 1$
- $P(A) \subseteq P(B)$ stands for the occurrence of A implies occurrence of B.
- $P(\bar{A})$ or $P(A')$ stands for the non-occurrence or negation of A.
- $P(A \cup B)$ stands for the occurrence of atleast one of A and B.
- $P(A \cap B)$ stands for simultaneous occurrence of A and B.
- $P(A' \cap B')$ stands for non-occurrence of both A and B.

Odds in Favour and Odds against of an Event

Let S be the sample space and E be an event. Let E' denotes the complement of event E. Then,

(i) odds in favour of event E

$$= \frac{\text{Number of cases favourable to } E}{\text{Number of cases against to } E}$$

$$= \frac{n(E)}{n(E')} = \frac{n(E)/n(S)}{n(E')/n(S)} = \frac{P(E)}{P(E')}$$

(ii) odds against of an event E

$$= \frac{\text{Number of cases against to } E}{\text{Number of cases favourable to } E}$$

$$= \frac{n(E')}{n(E)} = \frac{n(E')/n(S)}{n(E)/n(S)}$$

$$= \frac{P(E')}{P(E)}$$

Addition Theorem of Probability

If A and B are any two events in a sample space S, then the probability of occurrence of atleast one of the events A and B is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In general, $P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i)$

$$- \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$- \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)$$

Note For mutually exclusive events

(i) $P(A \cup B) = P(A) + P(B)$

(ii) $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

MIND Shakers

Which mathematical symbols replaces the question mark to make the equation true?

18 ? 12 ? 6 ? 4 = 40

REMEMBER

- If A, B and C are three events in a sample space S, then
  $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
  $- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
- If A and B are any two events, then
  (i) $P(A) = P(A - B) + P(A \cap B) = P(A \cap B') + P(A \cap B)$
  (ii) $P(B) = P(B - A) + P(A \cap B) = P(B \cap A') + P(A \cap B)$
- P (Atleast two of A, B and C occur)
  $= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$
- P (Exactly two of A, B and C occur)
  $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$
- P (Exactly one of A, B and C occurs)
  $= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A)$
  $- 2P(A \cap B) + 3P(A \cap B \cap C)$

SPEED BOOSTER

- When three dice are thrown, let n $(3 \le n \le 18)$ be the sum of numbers on them, then number of ways of getting the sum n is given by

$$\begin{cases} \frac{(n-1)(n-2)}{2}, & \text{if } 3 \le n \le 8 \\ \frac{(19-n)(20-n)}{2}, & \text{if } 13 \le n \le 18 \\ 25, & \text{if } (n = 9, 12) \\ 27, & \text{if } (n = 10, 11) \end{cases}$$

- If there are n letters corresponding to n envelopes are placed in the envelopes at random, then

  (i) probability that all letters are in right envelopes $= \frac{1}{n!}$

  (ii) probability that all letters are not in right envelopes

  $$= 1 - \frac{1}{n!}$$

  (iii) probability that no letter is in right envelopes

  $$= \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}\right]$$

  (iv) probability that exactly r letters are in right envelopes

  $$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r} \frac{1}{(n-r)!}\right]$$

- **Selection of shoes from a cupboard**

  (i) Out of n pair of shoes, if k shoes are selected at random, then the probability that there is no pair, is $\frac{{}^{n}C_{k}\, 2^{k}}{{}^{2n}C_{k}}$.

  (ii) The probability that there is atleast one pair, is $(1 - p)$.

- **Selection of squares from the chessboard** If r squares are selected from a chessboard, then probability that they lie on a diagonal, is

$$\frac{4({}^{7}C_{r} + {}^{6}C_{r} + {}^{5}C_{r} + \ldots + {}^{1}C_{r}) + 2 \cdot {}^{8}C_{r}}{{}^{64}C_{r}}$$

- Let A be a set containing n elements. Then, probability that

  (i) r subset of A, selected at random have no common elements is $\left(\frac{2^r - 1}{2^r}\right)^n$.

(ii) Two subsets of A selected at random will have r elements in common is ${}^{n}C_{r}\,\frac{3^{n-r}}{4^{n}}$.

(iii) If B and C are two subsets of A, then probability that $C \cup B = A$ is $\left(\frac{3}{4}\right)^{m}$.

- If A and B are two finite sets and if a mapping is selected at random from the set of all mapping from A into B, then the probability that the mapping is

(i) an one-one function $= \frac{{}^{n(B)}P_{n(A)}}{n(B)^{n(A)}}$

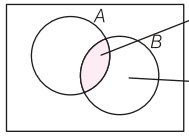
(ii) a many-one function $= 1 - \frac{{}^{n(B)}P_{n(A)}}{n(B)^{n(A)}}$

(iii) a constant function $= \frac{n(B)}{n(B)^{n(A)}}$

(iv) an one-one and onto function $= \frac{n(A)!}{n(B)^{n(A)}}$

Conditional Probability

Let A and B be any two events, $B \neq \phi$, then $P\left(\frac{A}{B}\right)$ denotes the conditional probability of occurrence of event A when B has already occurred, i.e. $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$



Favourable part of $P(A/B)$, i.e. $P(A \cap B)$

Act as sample space for $P(A/B)$, i.e. $P(B)$

While finding $P\left(\frac{A}{B}\right)$, B work as sample space and $(A \cap B)$ work as the favourable events as shown in above figure.

Multiplication Theorem

If A and B are two events, then $P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right)$, where $B \neq \phi$ and $P\left(\frac{A}{B}\right)$ denotes probability of occurrence of A when B has already occurred.

Also, if A, B and C are three events, then

$$P(A \cap B \cap C) = P(A) \cdot P\left(\frac{B}{A}\right) \cdot P\left(\frac{C}{A \cap B}\right)$$

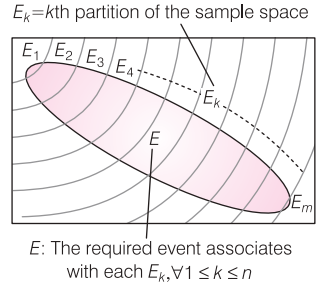
In general, $P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P\left(\frac{A_2}{A_1}\right) \cdot P\left(\frac{A_3}{A_1 \cap A_2}\right) \ldots P\left(\frac{A_n}{A_1 \cap A_2 \cap \ldots \cap A_{n-1}}\right)$

REMEMBER

- If two events are independent, then $P(A \cap B) = P(A) \cdot P(B)$
- If A, B and C are three independent events, then $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$
- If A and B are independent events, then $P(A \cup B) = 1 - P(A')\,P(B')$
- If A and B are independent events, then
  (i) A and B' are independent events.
  (ii) A' and B are independent events.
  (iii) A' and B' are independent events.
- $P(A/B) + P(\overline{A}/B) = 1$
- $P(B) = P(A) \cdot P(B/A) + P(\overline{A}) \cdot P\left(\frac{B}{\overline{A}}\right)$

Law of Total Probability

Let S be the sample space and $E_1, E_2, E_3, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If E is any event which occurs, if anyone of the events $E_1, E_2, \ldots, E_n$ occurs, then



$$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_3}\right) + \ldots + P(E_n) \cdot P\left(\frac{E}{E_n}\right)$$

or $P(E) = \sum_{i=1}^{n} P(E_i)\,P\left(\frac{E}{E_i}\right)$

Partition of a Set

Consider a sample space S. Let $E_1, E_2, \ldots, E_n$ be the set of mutually exclusive and exhaustive events of sample space S. These events are said to be partition of sample space S, if

$$E_i \cap E_j = \phi \text{ for } i \neq j, 1 \le i, j \le n \text{ and } \sum_{i=1}^{n} P(E_i) = 1$$

Baye's Theorem

If $E_1, E_2, E_3, \ldots, E_n$ are n mutually exclusive and exhaustive events and E is an event which occurs together with either of E_i, then

$$P\left(\frac{E_i}{E}\right) = \frac{P(E_i) \cdot P\left(\frac{E}{E_i}\right)}{\sum_{i=1}^{n} P(E_i) P\left(\frac{E}{E_i}\right)}, \text{ where } i = 1, 2, 3, \ldots, n.$$

Note (i) If $P(E_1) = P(E_2) = ... = P(E_n)$, then by Baye's theorem

$$P\left(\frac{E_k}{E}\right) = \frac{P\left(\frac{E}{E_k}\right)}{P\left(\frac{E}{E_1}\right) + P\left(\frac{E}{E_2}\right) + ... + P\left(\frac{E}{E_n}\right)}$$

(ii) If $E_1, E_2, E_3, ..., E_n$ form a partition of an event E, then

$$P\left(\frac{E_k}{E}\right) = \frac{P(E_k)}{P(E_1) + P(E_2) + ... + P(E_n)}$$

(iii) $P\left(\frac{E}{E_i}\right)$ gives the contribution of E_i in the occurrence of E.

(iv) The probabilities $P(E_1), P(E_2), ..., P(E_n)$ which are known before the experiment takes place are called a priori probabilities and $P\left(\frac{E_i}{E}\right)$ are called posteriori probabilities.

Concept for Asserting the Truth Statement

If P_1 and P_2 are the probabilities of speaking the truth of two independent witnesses A and B, then

(i) P (their combined statement is true)

$$= \frac{P_1 P_2}{P_1 P_2 + (1 - P_1)(1 - P_2)}$$

Note In this case, it has been assumed that we have no knowledge of the event except the statement made by A and B.

(ii) If P is the probability of happening of the event before their statement, then

P (their combined statement is true)

$$= \frac{PP_1P_2}{PP_1P_2 + (1 - P)(1 - P_1)(1 - P_2)}$$

Here, it has been assumed that the statement given by all the independent witness can be given by two ways only, so that if all the witness tell falsehood, they agree in telling the same falsehood. If this is not the case and c is the chance of their coincidence, then

Probability that the statement is true

$$= P_1\ P_2\ P$$

Probability that the statement is false

$$= (1 - P_1)(1 - P_2)(1 - P)\ c$$

MASTER STROKES

1. It has been found that if A and B play a game 12 times, A wins 6 times, B wins 4 times and they draw twice. A and B take part in a series of 3 games. The probability that they will win alternately is

(a) $\frac{5}{72}$ (b) $\frac{5}{36}$ (c) $\frac{19}{27}$ (d) None of these

2. A draws two cards at random from a pack of 52 cards. After returning them to the pack and shuffling it, B draws two cards at random. The probability that their draws contain exactly one common cards, is

(a) $\frac{25}{546}$ (b) $\frac{50}{663}$ (c) $\frac{25}{663}$ (d) None of these

3. All the heart cards are taken out from a pack of playing cards. From these cards, cards are drawn one-by-one without replacement till the ace of heart comes. The probability that ace comes in the 4th draw, is

(a) $\frac{1}{13}$ (b) $\frac{12}{13}$

(c) $\frac{4}{13}$ (d) None of these

4. If a problem in Mathematics is given to three students A, B and C and their respective probability of solving the problem are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. Then, the probability that the problem is solved, is

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

5. If $P(B) = \frac{3}{4}$, $P(A \cap B \cap \overline{C}) = \frac{1}{3}$ and $P(\overline{A} \cap B \cap \overline{C}) = \frac{1}{3}$, then $P(B \cap C)$ is equal to

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{1}{15}$ (d) $\frac{1}{9}$

6. The probability that A speaks the truth is 4/5, while this probability for B is 3/4. The probability that they contradict each other when asked to speak on a fact, is

(a) $\frac{4}{5}$ (b) $\frac{1}{5}$ (c) $\frac{7}{20}$ (d) $\frac{3}{20}$

7. The probability that in a year of 22nd century choose at random, there will be 53 Sunday, is

(a) $\frac{3}{28}$ (b) $\frac{2}{28}$ (c) $\frac{7}{28}$ (d) $\frac{5}{28}$

8. A fair die is rolled. The probability that the first time 1 occurs at the even throw, is

(a) $\frac{1}{6}$ (b) $\frac{5}{11}$ (c) $\frac{6}{11}$ (d) $\frac{5}{36}$

9. There are n urns each containing $(n + 1)$ balls such that ith urn contains i white balls and $(n + 1 - i)$ red balls. Let U_i be the event of selecting ith urn, $i = 1, 2, 3, ..., n$ and W denotes the event of getting a white ball. If $P(U_i) \propto i$, where $i = 1, 2, 3, ..., n$, then $\lim_{n \to \infty} P(W)$ is equal to

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

10. One Indian and four American men and their wives are to be seated randomly around a circular table. The conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife, is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

11. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent, is

(a) 2, 4 or 8 (b) 3, 6 or 9 (c) 4 or 8 (d) 5 or 10

12. A fair die is tossed repeatedly until a six is obtained. If X denotes the number of tosses required, then the probability that $X = 3$ equals

(a) $\frac{25}{216}$ (b) $\frac{25}{36}$ (c) $\frac{5}{36}$ (d) $\frac{125}{216}$

13. Let E' denotes the complement of an event E. If E, F and G be pairwise independent events such that $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then, $P(E' \cap F'/G)$ equals

(a) $P(E') + P(F')$ (b) $P(E') - P(F')$
(c) $P(E') - P(F)$ (d) $P(E) - P(F')$

14. A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is 3/4. If the signal received at station B is green, then the probability that the original signal was green, is

(a) $\frac{3}{5}$ (b) $\frac{6}{7}$ (c) $\frac{20}{23}$ (d) $\frac{9}{20}$

15. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denotes the probability of occurrence of the event T, then

(a) $P(E) = \frac{4}{5}, P(F) = \frac{3}{5}$ (b) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$
(c) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$ (d) $P(E) = \frac{6}{5}, P(F) = \frac{1}{5}$

16. Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls while U_2 contains only 1 white ball. A fair coin is tossed. If head appears, then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears, then 2 balls are drawn at random from U_1 and put into U_2. Now, 1 ball is drawn at random from U_2. Then, the probability of the drawn ball from U_2 being white is

(a) $\frac{13}{30}$ (b) $\frac{23}{30}$ (c) $\frac{19}{30}$ (d) $\frac{11}{30}$

17. If C and D are two events such that $C \subset D$ and $P(D) \neq 0$, then the correct statement among the following is

(a) $P\left(\frac{C}{D}\right) = P(C)$ (b) $P\left(\frac{C}{D}\right) \geq P(C)$
(c) $P\left(\frac{C}{D}\right) < P(C)$ (d) $P\left(\frac{C}{D}\right) = \frac{P(D)}{P(C)}$

18. Let A, B and C be the pairwise independent events with $P(C) > 0$ and $P(A \cap B \cap C) = 0$. Then, $P\left(A^C \cap \frac{B^C}{C}\right)$ is equal to

(a) $P(A^C) + P(B^C)$ (b) $P(A^C) - P(B^C)$
(c) $P(A^C) - P(B)$ (d) $P(A) - P(B^C)$

19. Four fair dice D_1, D_2, D_3 and D_4, each having six faces numbered 1, 2, 3, 4, 5 and 6 are rolled simultaneously. The probability that D_4 shows a number on one of D_1, D_2 and D_3, is

(a) $\frac{91}{216}$ (b) $\frac{108}{216}$ (c) $\frac{125}{216}$ (d) $\frac{127}{216}$

20. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational atleast two of its engines must function. Let X denotes the event that ship is operational and X_1, X_2 and X_3 denote respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is not true?

(a) $P\left(\frac{X_1^c}{X}\right) = \frac{7}{8}$

(b) P [Exactly two engines of the ship are functioning $/X$] $= \frac{7}{8}$

(c) $P\left(\frac{X}{X_2}\right) = \frac{5}{16}$

(d) $P\left(\frac{X}{X_1}\right) = \frac{7}{16}$

21. Let X and Y be two events such that $P\left(\frac{X}{Y}\right) = \frac{1}{2}$, $P\left(\frac{Y}{X}\right) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is incorrect?

(a) $P(X \cup Y) = \frac{2}{3}$ (b) X and Y are independent
(c) $P(X^c \cap Y) = \frac{1}{6}$ (d) X and Y are not independent

22. The probability that randomly selected calculator from a store is of brand r, is proportional to r, where $r = 1, 2, \ldots, 6$. If the probability of a calculator of brand r being defective is $\frac{7-r}{21}$. Then, the probability that a calculator randomly selected from the store being defective, is

(a) $\frac{8}{63}$ (b) $\frac{13}{63}$ (c) $\frac{55}{63}$ (d) $\frac{50}{63}$

To be Continued at Page 84

PROBABILITY DISTRIBUTION

Random Variable

Let S be a sample space associated with a given random experiment. A real valued function X defined on S as $X : S \to R$ is called a random variable.

There are two types of random variable.

Discrete Random Variable

If the range of the real function $X : S \to R$ is a finite set or an infinite set of real numbers, then it is called a discrete random variable.

Continuous Random Variable

If the range of X is an interval (a, b) of R, then X is called a continuous random variable.

Probability Distribution

If the random variable X takes n distinct values $x_1, x_2, x_3, \ldots, x_n$, then $\{X = x_1\}, \{X = x_2\}, \{X = x_3\}, \ldots, \{X = x_n\}$ are mutually exclusive and exhaustive events. Then, the system of distribution of numbers

X	x_1	x_2	x_3	...	x_n
$P(X)$	p_1	p_2	p_3	...	p_n

is said to be the probability distribution of the random variable X.

The expectation (mean), variance and standard deviation of random X is given below:

(i) The expectation (mean) of random variable X is

$$E(X) = \sum_{i=1}^{n} x_i\, p_i$$

(ii) **Variance** $\text{var}\,(X) = \sigma_x^2 = E(X^2) - \{E(X)\}^2$,

where $E(X^2) = \sum_{i=1}^{n} x_i^2 p_i$

(iii) **Standard deviation**

$$\sqrt{V(X)} = \sigma_x = \sqrt{E(X^2) - \{E(X)\}^2}$$

Note (i) If $Y = aX + b$, then

(a) $E(Y) = E(aX + b) = aE(X) + b$

(b) $\sigma_y^2 = V(Y) = a^2V(X) = a^2\sigma_x^2$

(c) $\sigma_y = \sqrt{V(Y)} = |a|\,\sigma_x$

(ii) If $Z = aX^2 + bX + c$, then

$E(Z) = E(aX^2 + bX + c) = aE(X^2) + bE(X) + c$

Binomial Distribution

Bernaulli Trial

In a random experiment, if there are any two events, "success and failure" and the sum of probabilities of these two events is 1. Then, the outcomes of such experiment is known as Bernaulli trial.

Binomial Distribution

The probability of getting r success in n trial of a random experiment is ${}^nC_r\, p^r q^{n-r}$, where p is probability of success and q is probability of failure in each trial, also $p + q = 1$.

Note

(i) If $p = q$, then probability of r successes in n trials is ${}^nC_r p^n$.

(ii) If the total number of trials is n in any attempt and if there are N such attempts, then the total number of r successes is $N({}^nC_r p^r q^{n-r})$.

(iii) Mean = $E(X) = np$

(iv) Variance = $\sigma_x^2 = npq$

(v) Standard deviation = $\sigma_x = \sqrt{npq}$

(vi) Mean is always greater than variance.

SPEED **BOOSTER**

- If a die has m faces marked with the numbers 1, 2, 3, ..., m and if such n dice are thrown, then the probability that the sum of the numbers exhibited on the upper face equal to p, is given by the coefficient of x^p in the expansion of $\dfrac{(x + x^2 + x^3 + \ldots + x^m)^n}{m^n}$.
- Let X and Y be random variables on the sample space S and K is a real number, then
  (a) $E(KX) = K \cdot E(X)$
  (b) $E(X + Y) = E(X) + E(Y)$

Geometrical Probability

If the total number of outcomes of a trial in a random experiment is infinite, in such cases the definition of probability is modified and the general expression for the probability of occurrence of an event is given by

$$P(E) = \frac{\text{Measures of specified part of the region}}{\text{Measure of whole region}}$$

where, 'measure' means length, area or volume of the region, if we are dealing with one, two or three dimensional space, respectively.

DO YOU Know?

There are 2598960 five-card hands possible in a 52-card deck of cards.

MASTER STROKES

1. A biased coin with probability p, $0 < p < 1$, of heads is tossed until a head appears for the first time. If the probability that the number of tosses required is even is 2/5, then p equals

(a) 1/3 *(b)* 2/3 *(c)* 2/5 *(d)* 3/5

2. If X follows a binomial distribution with the parameters $n = 8$ and $p = 1/2$, then $P(|X - 4| \le 2)$ equals

(a) $\frac{118}{128}$ *(b)* $\frac{119}{128}$ *(c)* $\frac{117}{128}$ *(d)* None of these

3. If X follows a binomial distribution with the parameters $n = 100$ and $p = 1/3$, then $P(X = r)$ is maximum, when r is equal to

(a) 32 *(b)* 34 *(c)* 33 *(d)* 31

4. A fair die is tossed eight times. The probability that a third six is observed in the eight throw, is

(a) $\frac{{}^7C_2 \times 5^5}{6^7}$ *(b)* $\frac{{}^7C_2 \times 5^5}{6^8}$ *(c)* $\frac{{}^7C_2 \times 5^5}{6^6}$ *(d)* None of these

5. Four digit numbers are formed using each of the digits 1, 2, …, 8 only once. One number from them is picked up at random. The probability that the selected number contains unity, is

(a) 1/8 *(b)* 1/4 *(c)* 1/2 *(d)* None of these

6. A die is thrown $(2n + 1)$ times. The probability that faces with even numbers show odd number of times, is

(a) $\frac{1}{2}$ *(b)* $< \frac{1}{2}$ *(c)* $> \frac{1}{2}$ *(d)* None of these

7. A fair coin is tossed repeatedly. The probability of getting a result in fifth toss different from those obtained in the first four tosses, is

(a) 1/2 *(b)* 1/32 *(c)* 31/32 *(d)* 1/16

8. An ordinary die is rolled a certain number of times. If the probability of getting an odd number 2 times is equal to the probability of getting an even number 3 times, then the probability of getting an odd number of times is

(a) 1/32 *(b)* 5/16 *(c)* 1/2 *(d)* None of these

9. A coin is tossed $2n$ times. The chance that the number of times one gets head is not equal to the number of times one gets tail, is

(a) ${}^{2n}C_n\left(\frac{1}{2}\right)^{2n}$ *(b)* $1 - {}^{2n}C_n$

(c) $1 - {}^{2n}C_n\left(\frac{1}{4^n}\right)$ *(d)* None of these

10. A card is drawn from a pack of 52 playing cards. The card is replaced and the pack is reshuffled. If this is done six times, then the probability that 2 hearts, 2 diamonds and 2 black cards are drawn, is

(a) $90 \times \left(\frac{1}{4}\right)^6$ *(b)* $\frac{45}{2} \times \left(\frac{3}{4}\right)^4$ *(c)* $90 \times \left(\frac{1}{2}\right)^{10}$ *(d)* None of these

11. From a box containing 20 tickets marked with numbers 1 to 20 and four tickets are drawn one-by-one. After each draw, the ticket is replaced. The probability that the largest value of tickets drawn is 15, is

(a) $\left(\frac{3}{4}\right)^4$ *(b)* $\frac{27}{320}$ *(c)* $\frac{27}{1280}$ *(d)* None of these

12. A coin is tossed 3 times by 2 persons. The probability that both get equal number of heads, is

(a) 3/8 *(b)* 1/9 *(c)* 5/16 *(d)* None of these

13. If the mean and the variance of a binomial distribution are 4 and 2, respectively. Then, the probability of 2 successes is

(a) 28/256 *(b)* 219/256 *(c)* 128/256 *(d)* 37/256

14. If two coins are tossed five times, then the probability of getting 5 heads and 5 tails is

(a) 63/256 *(b)* 1/1024 *(c)* 2/205 *(d)* 9/64

15. If X and Y are independent binomial variates $B(5, 1/2)$ and $B(7, 1/2)$, then $P(X + Y) = 3$, is

(a) 35/47 *(b)* 55/1024 *(c)* 220/512 *(d)* 11/204

16. In a binomial distribution $B(n, p = 1/4)$. If the probability of atleast one success is greater than or equal to 9/10, then n is greater than

(a) $\frac{1}{\log_{10} 4 - \log_{10} 3}$ *(b)* $\frac{1}{\log_{10} 4 + \log_{10} 3}$

(c) $\frac{9}{\log_{10} 4 - \log_{10} 3}$ *(d)* $\frac{4}{\log_{10} 4 - \log_{10} 3}$

17. If two cards are drawn successively with replacement from a well shuffled deck of 52 cards, then the mean of the number of aces is

(a) 1/13 *(b)* 3/13 *(c)* 2/13 *(d)* None of these

18. A die is thrown 100 times, getting an even number is considered a success. The variance of the number of successes is

(a) 10 *(b)* 25 *(c)* 18 *(d)* 12

19. There are 12 white and 12 red balls in a bag. Balls are drawn one-by-one with replacement from the bag. The probability that 7th drawn ball is 4th white, is

(a) 1/4 *(b)* 5/32 *(c)* 3/16 *(d)* 5/16

20. A die is tossed thrice. If event of getting an even number is a success, then the probability of getting atleast two successes is

(a) 7/8 *(b)* 1/4 *(c)* 2/3 *(d)* 1/2

21. The mean and variance of a binomial distribution are 4 and 3, respectively. Then, the probability of getting exactly six successes in this distribution, is

(a) ${}^{16}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^{10}$ *(b)* ${}^{16}C_6\left(\frac{1}{4}\right)^{16}\left(\frac{3}{4}\right)^{20}$

(c) ${}^{16}C_6\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^{12}$ *(d)* ${}^{16}C_9\left(\frac{1}{4}\right)^{16}\left(\frac{3}{4}\right)^{20}$

22. The sum of the mean and variance of a binomial distribution is 15 and the sum of their squares is 117. The mean of the distribution is
(a) 6 *(b)* 9 *(c)* 3 *(d)* 12

23. Consider 5 independent Bernaulli's trials each with probability of success p. If the probability of atleast one failure is greater than or equal to $\frac{31}{32}$, then p lies in the interval
(a) $\left(\frac{1}{2}, \frac{3}{4}\right]$ *(b)* $\left(\frac{3}{4}, \frac{11}{12}\right]$ *(c)* $\left[0, \frac{1}{2}\right]$ *(d)* $\left(\frac{11}{12}, 1\right]$

24. A random variable X takes values $-1, 0, 1$ and 2 with probabilities $\frac{1+3p}{4}, \frac{1-p}{4}, \frac{1+2p}{4}$ and $\frac{1-4p}{4}$, respectively, where p varies over R. Then, the minimum and maximum values of the mean of X are respectively
(a) $-\frac{7}{4}$ and $\frac{1}{2}$ *(b)* $-\frac{1}{16}$ and $\frac{5}{16}$
(c) $-\frac{7}{4}$ and $\frac{5}{16}$ *(d)* $-\frac{1}{16}$ and $\frac{5}{4}$

25. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing, is
(a) $\frac{17}{3^5}$ *(b)* $\frac{13}{3^5}$ *(c)* $\frac{11}{3^5}$ *(d)* $\frac{10}{3^5}$

Answers

1. (a)	**2.** (b)	**3.** (c)	**4.** (b)	**5.** (c)
6. (a)	**7.** (d)	**8.** (c)	**9.** (c)	**10.** (c)
11. (b)	**12.** (c)	**13.** (a)	**14.** (a)	**15.** (b)
16. (a)	**17.** (c)	**18.** (b)	**19.** (b)	**20.** (d)
21. (a)	**22.** (b)	**23.** (c)	**24.** (d)	**25.** (c)

Continued from Page 81

23. If six students including two particular students A and B stand in a row, then the probability that A and B are separated with one student in between them, is
(a) $\frac{8}{15}$ *(b)* $\frac{1}{5}$ *(c)* $\frac{2}{15}$ *(d)* $\frac{4}{15}$

24. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}$ and $\frac{1}{8}$. Then, the probability that the problem is solved correctly by atleast one of them, is
(a) $\frac{235}{256}$ *(b)* $\frac{21}{256}$ *(c)* $\frac{3}{256}$ *(d)* $\frac{253}{256}$

25. A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls. If 1 ball is drawn from each of the three boxes B_1, B_2 and B_3, then the probability that all 3 drawn balls are of the same colour, is
(a) $\frac{82}{648}$ *(b)* $\frac{90}{648}$ *(c)* $\frac{558}{648}$ *(d)* $\frac{566}{648}$

26. A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, then the probability that these 2 balls are drawn from box B_2, is
(a) $\frac{116}{181}$ *(b)* $\frac{126}{181}$ *(c)* $\frac{65}{181}$ *(d)* $\frac{55}{181}$

27. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval $(0, 1)$. Then, the ratio of probability of occurrence of E_1 to probability of occurrence of E_3 is
(a) 3 *(b)* 2 *(c)* 6 *(d)* 4

28. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for the complement of the event A. Then, the events A and B are
(a) independent but not equally likely
(b) independent and equally likely
(c) mutually exclusive and independent
(d) equally likely but not independent

29. If the integers m and n are chosen at random between 1 and 100, then the probability that a number of the form $7^m + 7^n$ is divisible by 5, is
(a) $\frac{1}{5}$ *(b)* $\frac{1}{7}$ *(c)* $\frac{1}{4}$ *(d)* $\frac{1}{49}$

30. The minimum number of times a fair coin needs to be tossed, so that the probability of getting atleast two heads is atleast 0.96, is
(a) 7 *(b)* 9 *(c)* 8 *(d)* 5

Answers

1. (b)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (a)
6. (c)	**7.** (d)	**8.** (b)	**9.** (b)	**10.** (c)
11. (d)	**12.** (a)	**13.** (c)	**14.** (c)	**15.** (a)
16. (d)	**17.** (d)	**18.** (c)	**19.** (a)	**20.** (a)
21. (d)	**22.** (a)	**23.** (d)	**24.** (a)	**25.** (a)
26. (d)	**27.** (c)	**28.** (a)	**29.** (a)	**30.** (c)

BRAIN TEASERS

To boost up your rank in JEE Main and Advanced

1. If $(4+\sqrt{15})^{x^2-2x+1}+(4-\sqrt{15})^{x^2-2x-1}=\dfrac{2}{4-\sqrt{15}}$, then find the value of x.

2. Determine the real solutions of the inequality $\dfrac{4x^2}{(1-\sqrt{1+2x})^2}<2x+9$.

3. Find the total number of terms in the expansion of $(a+b+c+d)^n$, $n\in N$.

4. If $(1+x+2x^2)^{20}=a_0+a_1x+a_2x^2+\ldots+a_{40}x^{40}$, then find the value of $a_0+a_2+a_4+\ldots+a_{38}$.

5. Find the sum of the infinite series $\dfrac{5}{13}+\dfrac{55}{(13)^2}+\dfrac{555}{(13)^3}+\dfrac{5555}{(13)^4}+\ldots\infty$.

6. Determine the smallest possible positive integer x, whose last decimal digit is 6 and if we erase this last 6 and put it in front of the remaining digits, we get four times x.

7. Prove that there are infinitely many positive integers a such that $x=n^4+a$ is not a prime for any positive integer n.

8. If $(x+1)(x+2)(x+3)(x+4)=24$, then find the value of x.

9. Determine all possible values of

$$S=\frac{x_1}{x_1+x_2+x_4}+\frac{x_2}{x_1+x_2+x_3}+\frac{x_3}{x_2+x_3+x_4}+\frac{x_4}{x_1+x_3+x_4}$$

for arbitrary positive real numbers x_1, x_2, x_3 and x_4.

10. If a, b and c are positive real numbers, then find the greatest value of $a^2b^3c^4$, if $a+b+c=18$.

Solutions

1. Since, $4-\sqrt{15}=\dfrac{1}{4+\sqrt{15}}$

Then, given equation can be expressed as

$$(4+\sqrt{15})^{x^2-2x+1}+\frac{1}{(4+\sqrt{15})^{x^2-2x-1}}=2(4+\sqrt{15})$$

Put $y=(4+\sqrt{15})^{x^2-2x+1}$

Then, above equation can be expressed as

$$y+\frac{(4+\sqrt{15})^2}{y}=2(4+\sqrt{15})$$

$$\Rightarrow\quad y^2-2(4+\sqrt{15})y+(4+\sqrt{15})^2=0$$

$$\Rightarrow\quad \{y-(4+\sqrt{15})\}^2=0$$

$$\Rightarrow\quad y=4+\sqrt{15}$$

$$\Rightarrow\quad (4+\sqrt{15})^{x^2-2x+1}=(4+\sqrt{15})$$

$$\Rightarrow\quad x^2-2x+1=1$$

$$\Rightarrow\quad x(x-2)=0$$

$$\Rightarrow\quad x=0 \text{ or } 2$$

2. We have, $\dfrac{4x^2}{(1-\sqrt{1+2x})^2}<2x+9$

The expression on the left hand side is defined, if its denominator is non-zero, i.e. if $x\neq 0$; moreover if the expression under the square root is non-negative, i.e. $1+2x\geq 0\Rightarrow x\geq -\dfrac{1}{2}$

Now, multiply both the numerator and the denominator of the fraction by $(1+\sqrt{1+2x})^2$, we get

$$\frac{4x^2(1+\sqrt{1+2x})^2}{4x^2}<2x+9$$

$$\Rightarrow\quad (1+1+2x+2\sqrt{1+2x})<2x+9$$

$$\Rightarrow\quad 2\sqrt{1+2x}<7$$

$$\Rightarrow\quad 4+8x<49$$

$$\Rightarrow\quad x<\frac{45}{8}$$

$\therefore$ Required solution sets of the given inequality are

$$-\frac{1}{2} \le x < 0,\ 0 < x < \frac{45}{8}$$

3. We have, $(a + b + c + d)^n = [(a + b) + (c + d)]^n$

$$= (a + b)^n + {}^nC_1(a + b)^{n-1}(c + d) + {}^nC_2(a + b)^{n-2}(c + d)^2 + \ldots + {}^nC_n(c + d)^n$$

Number of terms in RHS

$$= (n + 1) + n \cdot 2 + (n - 1) \cdot 3 + \ldots + (n + 1)$$

$$= \sum_{k=0}^{n}(n - k + 1)(k + 1)$$

$$= \sum_{k=0}^{n}(n + 1) + \sum_{k=0}^{n}nk - \sum_{k=0}^{n}k^2$$

$$= \sum_{k=0}^{n}(n + 1) + \sum_{k=0}^{n}nk - \sum_{k=0}^{n}k^2$$

$$= (n + 1)\sum_{k=0}^{n}1 + n\sum_{k=0}^{n}k - \sum_{k=0}^{n}k^2$$

$$= (n + 1)(n + 1) + \frac{n \cdot n(n + 1)}{2} - \frac{n(n + 1)(2n + 1)}{6}$$

$$= \frac{(n + 1)(n + 2)(n + 3)}{6} = {}^{n+3}C_3$$

4. Given, $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots + a_{38}x^{38} + a_{39}x^{39} + a_{40}x^{40}$

On putting $x = 1$ and $x = -1$, we get

$$4^{20} = a_0 + a_1 + a_2 + \ldots + a_{38} + a_{39} + a_{40} \quad \ldots(i)$$

and $2^{20} = a_0 - a_1 + a_2 - \ldots + a_{38} - a_{39} + a_{40} \quad \ldots(ii)$

On adding Eqs. (i) and (ii), we get

$$2^{20}(2^{20} + 1) = 2(a_0 + a_2 + \ldots + a_{38} + a_{40})$$

$\therefore \quad 2^{19}(2^{20} + 1) = a_0 + a_2 + \ldots + a_{38} + a_{40}$

$\Rightarrow \quad a_0 + a_2 + \ldots + a_{38} = 2^{19}(2^{20} + 1) - a_{40} \quad \ldots(iii)$

But a_{40} = Coefficient of x^{40} in $(1 + x + 2x^2)^{20}$

= Coefficient of x^{40} in $[1 + x(1 + 2x)]^{20}$

General term of $[1 + x(1 + 2x)]^{20}$ is

$$T_{r+1} = {}^{20}C_r\, x^r(1 + 2x)^r$$

Put $r = 20$

$\therefore \quad T_{21} = {}^{20}C_{20}\, x^{20}(1 + 2x)^{20}$

$$= x^{20}[1 + {}^{20}C_1(2x)^1 + \ldots + {}^{20}C_{20}(2x)^{20}]$$

$\therefore$ Coefficient of $x^{40} = 2^{20}$

$\therefore \quad a_{40} = 2^{20}$

From Eq. (iii),

$$a_0 + a_2 + \ldots + a_{38} = 2^{19}(2^{20} + 1) - 2^{20} = 2^{19}(2^{20} - 1)$$

5. Let $S = \frac{5}{13} + \frac{55}{(13)^2} + \frac{555}{(13)^3} + \frac{5555}{(13)^4} + \ldots \infty$

$$\Rightarrow \quad \frac{S}{13} = \frac{5}{(13)^2} + \frac{55}{(13)^3} + \frac{555}{(13)^4} + \frac{5555}{(13)^5} + \ldots \infty$$

$$\therefore \quad \left(S - \frac{S}{13}\right) = \frac{5}{13} + \frac{50}{(13)^2} + \frac{500}{(13)^3} + \frac{5000}{(13)^4} + \ldots \infty$$

$$\Rightarrow S\left(1 - \frac{1}{13}\right) = \frac{5}{13} + \frac{5}{13}\left(\frac{10}{13}\right) + \frac{5}{13}\left(\frac{10}{13}\right)^2 + \frac{5}{13}\left(\frac{10}{13}\right)^3 + \ldots \infty$$

$$\Rightarrow \quad \frac{12}{13}S = \frac{5/13}{3/13} \qquad \left[\because S_\infty = \frac{a}{1 - r}\right]$$

$$\Rightarrow \quad \frac{12}{13}S = \frac{5}{3} \quad \Rightarrow \quad S = \frac{65}{36}$$

6. Let the digits of the original n-digit number x denoted by $a_{n-1}, a_{n-2}, \ldots, a_1, a_0 = 6$.

Then, $x = 10^{n-1}a_{n-1} + 10^{n-2}a_{n-2} + \ldots + 10a_1 + 6$

According to the question,

$$4x = 6 \cdot 10^{n-1} + (10^{n-2}a_{n-1} + 10^{n-3}a_{n-2} + \ldots + 10a_2 + a_1)$$

The expression in the parenthesis clearly equals to $\frac{x - 6}{10}$, therefore,

$$4x = 6 \cdot 10^{n-1} + \frac{x - 6}{10}$$

$$\Rightarrow \quad 40x = 6 \cdot 10^n + (x - 6)$$

$$\Rightarrow \quad 13x = 2(10^n - 1)$$

This shows that $10^n - 1$ (a number containing only 9) must be divisible by 13, by inspecting these numbers, we get 999999 is the first number which is divisible by 13.

$\therefore \quad 999999 = 13 \times 76923$

Therefore, $\quad x = 2 \times 76923 = 153846$

7. Let $a = 4b^4$, where $b > 1$ is an arbitrary integer. We shall to show that x is not prime.

Then, $\quad x = n^4 + 4b^4 + 4n^2b^4 - 4n^2b^2$

$$\Rightarrow \quad x = (n^2 + 2b^2)^2 - (2nb)^2$$

$$\Rightarrow \quad x = (n^2 + 2b^2 + 2nb)(n^2 + 2b^2 - 2nb)$$

$$\Rightarrow \quad x = ((n + b)^2 + b^2)((n - b)^2 + b^2)$$

Since $b > 1$, therefore both factors are greater than 1.

Hence, x is not prime. **Hence proved.**

8. We have, $(x + 1)(x + 2)(x + 3)(x + 4) = 24$

$$\Rightarrow \quad [(x + 1)(x + 4)][(x + 2)(x + 3)] = 24$$

$$\Rightarrow \quad (x^2 + 5x + 4)(x^2 + 5x + 6) = 24$$

Let $\quad x^2 + 5x = y$

$\therefore \quad (y + 4)(y + 6) = 24$

$\Rightarrow \quad y^2 + 10y = 0$

$\Rightarrow \quad y = 0$ or $y = -10$

Now, $\quad y = x^2 + 5x$

When $\quad y = 0$, then $x^2 + 5x = 0$

$\Rightarrow \quad x = 0, -5$

Again, when $y = -10$, then $x^2 + 5x + 10 = 0$

$\therefore \quad D = b^2 - 4ac = 25 - 4 \times 1 \times 10 = -15$

Since, $D < 0$, therefore, the equation $x^2 + 5x + 10 = 0$ has no real solution.

$\therefore \quad x = 0, -5$

9. We have, $S = \frac{x_1}{x_1 + x_2 + x_4} + \frac{x_2}{x_1 + x_2 + x_3} + \frac{x_3}{x_2 + x_3 + x_4} + \frac{x_4}{x_1 + x_3 + x_4}$

We decrease the sum S, if we increase the denominators of fraction to $x_1 + x_2 + x_3 + x_4$.

$$\therefore\ S > \frac{x_1}{x_1+x_2+x_3+x_4} + \frac{x_2}{x_1+x_2+x_3+x_4} + \frac{x_3}{x_1+x_2+x_3+x_4} + \frac{x_4}{x_1+x_2+x_3+x_4} = 1$$

We increase S, if we decrease the denominators of the first two fractions to $x_1 + x_2$ and other two to $x_3 + x_4$.

$$\therefore\quad S < \frac{x_1}{x_1+x_2} + \frac{x_2}{x_1+x_2} + \frac{x_3}{x_3+x_4} + \frac{x_4}{x_3+x_4} = 2$$

Now, we show that S attains every value in (1, 2) so this its range.

First, let $x_1 = x_2 = x$ and $x_3 = x_4 = 1$

Then, $$S = f(x) = \frac{2x}{2x+1} + \frac{2}{x+2}$$

$$\lim_{x \to 0^+} f(x) = 1 \text{ and } f(1) = \frac{4}{3}$$

As f is continuous in [0, 1], it attains every λ, where $1 < \lambda \le \frac{4}{3}$.

Let $x_1 = x_3 = x$ and $x_2 = x_4 = 1$, then

$$S = g(x) = \frac{2x}{x+2} + \frac{2}{2x+1}$$

$$\therefore\quad g(1) = \frac{4}{3} \text{ and } \lim_{x \to 0^+} g(x) = 2$$

Since, $g(x)$ is continuous on [0, 1], hence it attains every λ, where $\frac{4}{3} \le \lambda < 2$.

$\therefore$ The range of the sum S is (1, 2).

10. $a^2b^3c^4$ is greatest when $\left(\frac{a}{2}\right)^2\left(\frac{b}{3}\right)^3\left(\frac{c}{4}\right)^4$ is greatest.

Sum of all the factors is $2\left(\frac{a}{2}\right) + 3\left(\frac{b}{3}\right) + 4\left(\frac{c}{4}\right) = 18$

$\therefore$ Product of all these factors is greatest when all the factors are equal.

i.e. $$\frac{a}{2} = \frac{b}{3} = \frac{c}{4} = \lambda \quad \text{[say]}$$

$\Rightarrow\quad a = 2\lambda,\ b = 3\lambda \text{ and } c = 4\lambda$

$\because\quad a + b + c = 18$

$\Rightarrow\quad \lambda = 2$

$\therefore\quad a = 4,\ b = 6$

and $\quad c = 8$

Hence, greatest value of $a^2b^3c^4$

$= (4)^2(6)^3(8)^4$

Continued from Page 57

$$\begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = e$$

$\Rightarrow e = 0$ [$\because$ determinant of odd order skew-symmetric matrix is zero]

C. $$\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\gamma) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\alpha-\alpha) & \cos(\gamma-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & \cos(\beta-\beta) & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & \cos(\gamma-\gamma) \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$= 0 \times 0 = 0$

D. $$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{vmatrix} = 1(28-30) - 0 - 1(18-0)$$

$= -20$

$$\therefore\quad a_{23} = \frac{\text{Cofactor of } 6}{|A|} = \frac{-8}{-20} = \frac{2}{5}$$

30. A $\to$ q; B $\to$ p; C $\to$ s; D $\to$ r

A. Given, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar vectors.

$\Rightarrow\quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$ ...(i)

Since, $[\mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= 0$ [from Eq. (i)]

$\Rightarrow\ \mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}$ are also coplanar vectors.

B. $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 1$

$\Rightarrow |\mathbf{a}\times\mathbf{b}|\cdot|\mathbf{c}\times\mathbf{d}|\cos\theta = 1$

where θ is the angle between $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$.

$\Rightarrow \sin\alpha\sin\beta\cos\theta = 1$

where α is the angle between $\mathbf{a}$ and $\mathbf{b}$ and β is the angle between $\mathbf{c}$ and $\mathbf{d}$.

$\Rightarrow\quad \sin\alpha = 1, \sin\beta = 1, \cos\theta = 1$

$\Rightarrow\quad \alpha = 90^\circ, \beta = 90^\circ, \theta = 0^\circ$

$\Rightarrow \mathbf{a} \perp \mathbf{b}, \mathbf{c} \perp \mathbf{d}$ and normal to plane of $\mathbf{a}$ and $\mathbf{b}$ is also normal to plane of $\mathbf{c}$ and $\mathbf{d}$.

$\Rightarrow\ \mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are coplanar.

C. Given, $\mathbf{a} + \mathbf{b} + \sqrt{3}\,\mathbf{c} = 0$

$\Rightarrow\quad \mathbf{a} + \mathbf{b} = -\sqrt{3}\,\mathbf{c}$

$\Rightarrow\quad |\mathbf{a}+\mathbf{b}|^2 = |-\sqrt{3}\mathbf{c}|^2$

$\Rightarrow\ \mathbf{a}^2 + \mathbf{b}^2 + 2(\mathbf{a}\cdot\mathbf{b}) = 3\mathbf{c}^2$

$\Rightarrow\quad 1^2 + 1^2 + 2\cos\theta = 3(1^2)$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

$\Rightarrow\quad \cos\theta = \frac{1}{2}$

$\Rightarrow\quad \theta = \frac{\pi}{3}$

D. $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}+2\mathbf{b}|$

$\Rightarrow\ a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} = a^2 + 4b^2 + 4\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow\quad 2\mathbf{a}\cdot\mathbf{b} = -3b^2 < 0$

Hence, angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

TARGET CBSE 2017

TEST DRIVE

Comprehensive Simulator Test Series for XII Board Exam.

BOARD EXAM

(PAPER ONE PREP UP)

Typical Questions with Crispy Solutions

FULL TEST 5

(COMPLETE SYLLABUS) WITH SOLUTIONS

INSTRUCTIONS

- Question Numbers 1-4 in **Section A** carrying 1 mark each.
- Question Numbers 5-12 in **Section B** carrying 2 marks each.
- Question Numbers 13-23 in **Section C** carrying 4 marks each.
- Question Numbers 24-29 in **Section D** carrying 6 marks each.

Section A

1. Show that the relation R in the set $\{11, 12, 13\}$ given by $R = \{(11, 11), (12, 12), (13, 13), (11, 12), (12, 13)\}$ is reflexive.

2. A woman buys 8 dozen of bananas, 10 dozen of mangoes and 4 dozen of apples. Bananas cost ₹ 18 per dozen, mangoes ₹ 9 per dozen and apples ₹ 6 per dozen. Represent the quantities bought by a row matrix and price by a column matrix and hence obtain the total cost.

3. Define a binary operation '*' on the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $a * b = ab \pmod 6$. Show that 1 is the identity for '*', 1 and 5 are the only invertible elements with $1^{-1} = 1$ and $5^{-1} = 5$.

4. If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}}$ are non-zero vectors, then prove that they are parallel if and only if $a_1b_2 - a_2b_1 = 0$.

Section B

5. Prove that $\cos^{-1}\frac{5}{\sqrt{26}} = \frac{1}{4}\tan^{-1}\frac{120}{119}$.

6. Using the properties of determinants, prove that

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix} = 6x^2(1-x^2).$$

7. If $y = \{x + \sqrt{x^2+1}\}^m$, then prove that $(x^2+1)\,y_2 + xy_1 - m^2y = 0$.

8. Amongst all pairs of positive numbers with product 324, find those whose sum is the least.

9. Evaluate $\int \sqrt{(x-3)(5-x)}\,dx$.

10. Solve the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = e^x;\ x > 0.$$

11. Show that the vectors $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ are linearly independent.

12. If A and B are independent events and $P(A \cap B) = \frac{1}{8}, P(A' \cap B') = \frac{3}{8}$, then find $2P(A) + 4P(B)$, if $P(A) > P(B)$

Section C

13. Find the length and foot of the perpendicular from the point (1, 1, 2) to the plane $\mathbf{r}\cdot(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) + 5 = 0$.

14. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

15. If $y = \frac{\sqrt{a^2+x^2} + \sqrt{a^2-x^2}}{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}$, then show that

$$\frac{dy}{dx} = -\frac{2a^2}{x^3}\left[1 + \frac{a^2}{\sqrt{a^4-x^4}}\right].$$

16. Show that the normal at any point θ to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is at a constant distance from the origin.

17. If $y = \dfrac{ax - b}{(x-1)(x-4)}$ has a turning point $P(2, -1)$, then find the values of a and b and show that y is maximum at P.

18. Evaluate $\int \dfrac{dx}{x\{6(\log x)^2 + 7\log x + 2\}}$.

19. Find the equation of a curve passing through the point (0, 2), if the sum of the coordinates of any point on the curve exceeds the slope of the tangent to the curve at the point by 5.

20. If $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, $\mathbf{b} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, then check whether $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$ is equal to $\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ or not.

21. Find the cartesian as well as the vector equation of the planes passing through the intersection of the planes $\mathbf{r} \cdot (2\hat{\mathbf{i}} + 6\hat{\mathbf{j}}) + 12 = 0$ and $\mathbf{r} \cdot (3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 0$, which are at unit distance from the origin.

22. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers and their probabilities of accidents are 0.01, 0.03 and 0.15, respectively. One of the insured persons meets with an accident. Find the probability that he is a scooter driver.

23. Out of a group of 30 honest people, 20 always speak the truth. Two persons are selected at random from the group. Find the probability distribution of the number of selected persons who speak the truth. Also, find the mean of the distribution. What values are described in this question?

Section D

24. If the function $f : R \to R$ is defined by $f(x) = 2x - 3$ and $g : R \to R$ by $g(x) = x^3 + 5$, then find the value of $(fog)^{-1}(x)$.

25. Solve the following system of equations, using matrices.
$\dfrac{2}{x} + \dfrac{3}{y} + \dfrac{10}{z} = 4, \dfrac{4}{x} - \dfrac{6}{y} + \dfrac{5}{z} = 1, \dfrac{6}{x} + \dfrac{9}{y} - \dfrac{20}{z} = 2; x, y, z \neq 0$

26. Sketch and find the area bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$.

27. Evaluate $\int_0^{\pi/2} \dfrac{\cos x}{1 + \cos x + \sin x}\, dx$.

28. A housewife wishes to mix up two kinds of foods X and Y in such a way that mixture contains atleast 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents in 1 kg of food X and 1 kg of food Y are as given in the following table :

Food	Vitamin A	Vitamin B	Vitamin C
X	1	2	3
Y	2	2	1

If one kg of food X costs ₹ 6 and one kg of food Y costs ₹ 10, then find the least cost of the mixture which will produce the desired diet.

29. Find the equation of the plane through the points $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$.

Answers with Explanation

1. Here, 11, 12, 13 $\in$ set and (11, 11), (12, 12), (13, 13) $\in R$.
Thus, $(a, a) \in R, \forall\ a \in$ set. Hence, R is reflexive.

2. The quantity of fruits bought by the woman can be written in the form of a row matrix C given by $C = [8\ \ 10\ \ 4]$.
The corresponding prices per dozen of fruits can be written in the form of a column matrix D given by $D = \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$

$\therefore$ The total cost (in ₹) is given by matrix CD

$$= [8\ \ 10\ \ 4]\begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix} = [8 \times 18 + 10 \times 9 + 4 \times 6] = 258$$

3. We have, $1 * 0 = 0 = 0 * 1$, $\quad 1 * 1 = 1 = 1 * 1$,
$1 * 2 = 2 = 2 * 1$, $\quad 1 * 3 = 3 = 3 * 1$,
$1 * 4 = 4 = 4 * 1$, $\quad 1 * 5 = 5 = 5 * 1$
That is $x * 1 = x = 1 * x, \forall\ x \in A$
So, 1 is the identity element for '*' in A.
We have, $1 * 1 = 1 = 1 * 1$
$\therefore$ 1 is invertible and $1^{-1} = 1$
Also, $5 * 5 =$ Remainder when 25 is divided by $6 = 1$
$\therefore$ 5 is invertible and $5^{-1} = 5$

4. We have, $\mathbf{a} = \lambda\mathbf{b}$.

$\Rightarrow \quad a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} = \lambda(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}}) \Rightarrow a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} = \lambda b_1\hat{\mathbf{i}} + \lambda b_2\hat{\mathbf{j}}$

$\Rightarrow \quad a_1 = b_1\lambda$ and $a_2 = b_2\lambda \ \Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

$\Rightarrow \quad a_1b_2 - a_2b_1 = 0$ **Hence proved.**

5. LHS $= \cos^{-1}\dfrac{5}{\sqrt{26}}$

$$= \tan^{-1}\left[\frac{\sqrt{1 - \left(\frac{5}{\sqrt{26}}\right)^2}}{\frac{5}{\sqrt{26}}}\right]$$

$$= \tan^{-1}\frac{1}{5} = \frac{1}{2}\left[2\tan^{-1}\frac{1}{5}\right] = \frac{1}{2}\tan^{-1}\left[\frac{2 \cdot \frac{1}{5}}{1 - \left(\frac{1}{5}\right)^2}\right]$$

$$= \frac{1}{2}\tan^{-1}\frac{5}{12} = \frac{1}{4}\left(2\tan^{-1}\frac{5}{12}\right) = \frac{1}{4}\tan^{-1}\left[\frac{2\cdot\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2}\right]$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{120}{119}\right) = \text{RHS}$$ **Hence proved.**

6. $\text{LHS} = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix}$

$$\text{LHS} = x(x+1)(1-x)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & -(x-2) & -x \end{vmatrix}$$

$$= x(1-x^2)\begin{vmatrix} 0 & 0 & 1 \\ x & -1 & x \\ 4x & 2 & -x \end{vmatrix}$$

[applying $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$]

$= x(1-x^2)[1\cdot(2x+4x)]$ [expanding by R_1]

$= 6x^2(1-x^2) = \text{RHS}$ **Hence proved.**

7. We have,

$$y = \{x + \sqrt{x^2+1}\}^m \quad \ldots(\text{i})$$

$$\frac{dy}{dx} = m\{x+\sqrt{x^2+1}\}^{m-1}\frac{d}{dx}\{x+\sqrt{x^2+1}\}$$

$$= \frac{m\{x+\sqrt{x^2+1}\}^m}{\sqrt{x^2+1}} = \frac{my}{\sqrt{1+x^2}} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \quad y_1 = \frac{my}{\sqrt{1+x^2}} \Rightarrow y_1\sqrt{1+x^2} = my$$

$$\Rightarrow \quad y_1^2(1+x^2) = m^2y^2 \quad [\text{squaring both sides}]$$

$$\Rightarrow \quad 2y_1y_2(1+x^2) + y_1^2(2x) = 2m^2yy_1$$

[differentiating both sides w.r.t. x]

$$\Rightarrow \quad y_2(1+x^2) + xy_1 - m^2y = 0$$ **Hence proved.**

8. Let the required numbers be x and y.

Then, $xy = 324$ [given] …(i)

Let $S = x + y$, where S is the sum of the numbers.

$$\therefore \quad S = x + \frac{324}{x} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \quad \frac{dS}{dx} = 1 - \frac{324}{x^2} \quad \ldots(\text{ii})$$

$$\Rightarrow \quad \frac{d^2S}{dx^2} = \frac{648}{x^3} \quad \ldots(\text{iii})$$

Now, the critical points of S are given by $\frac{dS}{dx} = 0$.

$$\therefore \quad 1 - \frac{324}{x^2} = 0 \quad [\text{from Eq. (ii)}]$$

$$\Rightarrow \quad x = 18 \quad [\text{taking positive square root}]$$

Now, $\left(\frac{d^2S}{dx^2}\right)_{x=18} = \frac{648}{(18)^3}$ [from Eq. (iii)]

$$= \frac{1}{9} > 0$$

Thus, S is minimum when $x = 18$.

On putting $x = 18$ in Eq. (i), we get $y = 18$

Hence, the required numbers are both equal to 18.

9. Let $I = \int\sqrt{(x-3)(5-x)}\,dx$

$$= \int\sqrt{-x^2+8x-15}\,dx$$

$$= \int\sqrt{1^2-(x-4)^2}\,dx$$

$$= \frac{1}{2}(x-4)\sqrt{1^2-(x-4)^2} + \frac{1}{2}(1)^2\sin^{-1}\left(\frac{x-4}{1}\right) + C$$

$$= \frac{1}{2}(x-4)\sqrt{(x-3)(5-x)} + \frac{1}{2}\sin^{-1}(x-4) + C$$

10. Given, differential equation is

$$\frac{dy}{dx} + \frac{1}{x}y = e^x, (x>0) \quad \ldots(\text{i})$$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int\frac{1}{x}dx} = e^{\log x} = x$$

$\therefore$ The solution is given by

$$yx = \int xe^x dx + C$$

$$\Rightarrow \quad yx = e^x(x-1) + C$$

which is the required solution of given differential equation.

11. Let $\vec{\alpha} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \vec{\beta} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$

and $\vec{\gamma} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Let $x\vec{\alpha} + y\vec{\beta} + z\vec{\gamma} = 0$

$\therefore x(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + y(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}) + z(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 0$

$\Rightarrow (x+2y+3z)\hat{\mathbf{i}} + (-3x-4y+2z)\hat{\mathbf{j}} + (2x-y-z)\hat{\mathbf{k}} = 0$

On equating the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$$x + 2y + 3z = 0$$
$$-3x - 4y + 2z = 0$$
$$2x - y - z = 0$$

Now, $\begin{vmatrix} 1 & 2 & 3 \\ -3 & -4 & 2 \\ 2 & -1 & -1 \end{vmatrix}$

$= 1(4+2) - 2(3-4) + 3(3+8) = 41 \neq 0$

Hence, vectors $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are linearly independent.

Hence proved.

12. Let $P(A) = x$ and $P(B) = y$

Since, A and B are independent events.

$$\therefore \quad P(A \cap B) = P(A)\cdot P(B)$$

$$\Rightarrow \quad xy = \frac{1}{8} \quad [\because P(A\cap B) = 1/8] \ldots(\text{i})$$

and $P(A' \cap B') = \frac{3}{8}$ [given]

$$\Rightarrow \quad P(A')\cdot P(B') = \frac{3}{8} \Rightarrow (1-x)\cdot(1-y) = \frac{3}{8}$$

$$\Rightarrow \quad 1 - x - y + xy = \frac{3}{8}$$

$$\Rightarrow \quad x + y - xy = \frac{5}{8} \quad \ldots(\text{ii})$$

On solving Eqs. (i) and (ii), we get

$$x = \frac{1}{2} \text{ and } y = \frac{1}{4} \quad [\because x > y]$$

$$\therefore P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{4}$$

Hence, $2P(A) + 4P(B) = 2\cdot\frac{1}{2} + 4\cdot\frac{1}{4} = 2$

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INSTRUCTIONS

- Question Numbers 1-4 in **Section A** are Very Short Answers carrying 1 mark each.
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- Question Numbers 13-23 in **Section C** are Long Answers I carrying 4 marks each.
- Question Numbers 24-29 in **Section D** are Long Answers II carrying 6 marks each.

Section A

1. Evaluate $\begin{vmatrix} \sin 25^\circ & \cos 25^\circ \\ \sin 70^\circ & \cos 70^\circ \end{vmatrix}$.

2. Find the area included between the curve $y = \cos x$, if the X-axis and ordinates are $x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$.

3. Find the angle between the lines $\frac{x+4}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$.

4. Solve the differential equation $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$.

Section B

5. A population grows continuously at the rate of 8% per year. How long does it take for the population to double? Use differential equation for it.

6. If two adjacent sides of a parallelogram are $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$, then find the unit vectors along the diagonals of the parallelogram.

7. Without expanding, find the value of

$$\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}.$$

8. If $y = \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then prove that $\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$.

9. A binary operation * on the set {0, 1, 2, 3, 4, 5} is defined as : $a * b = \begin{cases} a+b \text{ if } & a+b<6 \\ a-b \text{ if } & a+b \geq 6 \end{cases}$

Show that zero is the identity element of this operation and each element 'a' of the set is invertible with $6 - a$, being the inverse of 'a'.

10. Find the matrix A such that $2A - 3B + 5C = 0$, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$.

11. Evaluate $\int \frac{\sin x}{\sqrt{1+\sin x}}\,dx$.

12. Let $f(x) = x^2 - 5x + 6$. Find $f(A)$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

Section C

13. In a bolt factory, machines A, B and C manufacture 25%, 35% and 40%, respectively of the total and of their output 5%, 4% and 2% are defective. A bolt is drawn at random and is found to be defective. What are the probabilities that it was manufactured by the machines A, B and C?

14. Let g be a real function defined as $g(x) = \sqrt{x-1}$. Find $(gogog)(x)$. Also, prove that $gog \neq g^2$.

15. If $x = \text{cosec}\,[\tan^{-1}\{\cos(\cot^{-1}(\sec(\sin^{-1}a)))\}]$ and $y = \sec[\cot^{-1}\{\sin(\tan^{-1}(\text{cosec}\,(\cos^{-1}a)))\}]$, where $a \in [0, 1]$, then find the relationship between x and y in terms of a.

16. Find the image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

17. The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times of the first number we get 12. Use determinants to find the numbers.

18. Show that
$\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$.

19. Find the area bounded by the curves $y = x$ and $y = x^3$.

20. If $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ and the vectors
$\mathbf{A} = \hat{\mathbf{i}} + x\hat{\mathbf{j}} + x^2\hat{\mathbf{k}}$, $\mathbf{B} = \hat{\mathbf{i}} + y\hat{\mathbf{j}} + y^2\hat{\mathbf{k}}$ and $\mathbf{C} = \hat{\mathbf{i}} + z\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ are non-coplanar, then find the value of $\frac{1}{3}xyz + 1$.

21. A variable plane is at a constant distance p from the origin and meets the coordinate axes in A, B and C. Show that the locus of the centroid of the tetrahedron $OABC$ is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

22. If the vertices P, Q and R of ΔPQR, having position vectors (1, 2, 3), (−1, 0, 0) and (0, 1, 2) respectively and then find $\angle PQR$ ($\angle PQR$ is the angle between the vectors PQ and QR). In PQR plane, a message is written ''clean the city''. What value(s) is indicate from this message?

23. If $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$, then differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$.

Section D

24. Every gram of wheat provides 0.1 g of proteins and 0.25 g of carbohydrates. The corresponding values of rice are 0.05 g and 0.5 g, respectively. Wheat costs ₹ 4 per kg and rice ₹ 6 per kg. The minimum daily requirements of proteins and carbohydrates for an average child are 50 g and 200 g, respectively. In what quantities should wheat and rice be mixed in the daily diet to provide minimum daily requirements of proteins and carbohydrates at minimum cost?

25. Show that the angle between the tangent at any point P and the line joining P to the origin O is the same at all points on the curve $\log(x^2 + y^2) = k\tan^{-1}\left(\frac{y}{x}\right)$.

26. Let d_1, d_2 and d_3 be three mutually exclusive diseases. Let $S = \{s_1, s_2, s_3, \ldots, s_6\}$ be the set of observable symptoms of these diseases, e.g. s_1 is the shortness of breath, s_2 is the loss of weight, s_3 is the fatigue etc. Suppose a random sample of 10000 patients contains 3200 patients with disease d_1, 3500 with disease d_2 and 3300 with disease d_3. Also, 3100 patients with disease d_1, 3300 with disease d_2 and 3000 disease d_3 show the symptoms. Knowing that the patient has symptom S, the doctor wishes to determine the patient's illness. On the basis of this information, what should the doctor conclude?

27. Experiments show that the rate of inversion of sugarcane in dilute solution is proportional to the concentration $y(t)$ of the unaltered solution. Suppose that the concentration is $\frac{1}{100}$ at $t = 0$ and $\frac{1}{300}$ at $t = 10$ h. Find $y(t)$.

28. Sketch the region common to the circle $x^2 + y^2 = 16$ and the parabola $x^2 = 6y$. Also, find the area of the region using integration.

29. Evaluate $\int_1^3 (x^2 + 5x)dx$ as limit sums.

Answers

1. $-\frac{1}{\sqrt{2}}$ **2.** 2 sq units

3. $\cos^{-1}\frac{8\sqrt{3}}{15}$ **4.** $\sin^{-1} y + \sin^{-1} x = c$

5. $\left(\frac{25}{2}\log 2\right)$yr **6.** $-\frac{1}{\sqrt{69}}\hat{\mathbf{i}} - \frac{2}{\sqrt{69}}\hat{\mathbf{j}} + \frac{8}{\sqrt{69}}\hat{\mathbf{k}}$

7. 0 **10.** $\begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$

11. $2\sqrt{2}\sin\left(\frac{x}{2} - \frac{\pi}{4}\right) + \sqrt{2}\log\left|\sec\left(\frac{\pi}{4} - \frac{x}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + C$

12. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ **13.** $\frac{25}{69}, \frac{28}{69}, \frac{16}{69}$

14. $\sqrt{\sqrt{\sqrt{x-1}-1}-1}$ **15.** $x^2 = y^2 = 3 - a^2$

16. (1, 0, 7)

17. 3, 1 and 2 **19.** $\frac{1}{2}$ sq unit

20. $\frac{2}{3}$

22. $\angle PQR = \cos^{-1}\frac{10}{\sqrt{102}}$, 'Environmental protection'

23. $\frac{-1}{2}$ **24.** ₹ 2.8

26. The doctor should conclude that the patient is most likely to have disease d_2.

27. $e^{\left(-\frac{1}{10}\log 3\right)t - 2}$

28. $\left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right)$ sq units

29. $\frac{86}{3}$

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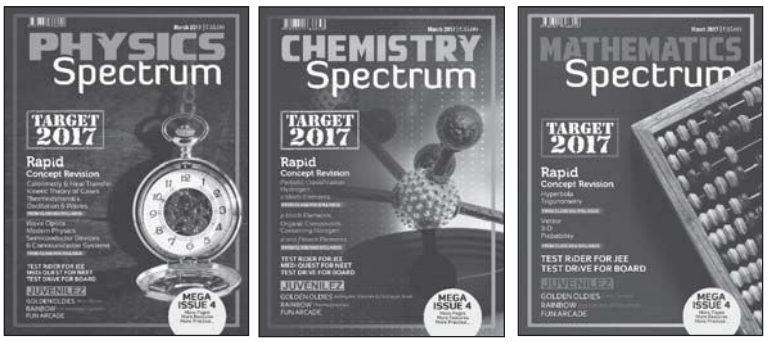

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FUN ARCADE

GODFREY HAROLD HARDY

Godfrey Harold "GH" Hardy was an English mathematician known for his achievements in number theory and mathematical analysis. Born on 7th February, 1877, in Surrey, England to teacher parents, he was the one who broke all British traditions of dwelling in "Applied Mathematics" and became a prominent figure in pure Mathematics.

In 1896, Hardy enrolled himself in Trinity college, Cambridge, where he took just two years to succeed in Mathematics Tripos Examination. In 1903, he received his MA which was arguably the topmost academic degree of the time.

In the year 1911, Hardy collaborated with fellow English mathematician. John E Littlewood in coming up with propounding mathematical analysis and analytic number theory. In prime number theory, they proved some conditional results and developed number theory as a system of conjectures.

Hardy also collaborated with Wilhelm Weinberg to come up with the Hardy-Weinberg principle, a basic principle of population genetics.

In 1914, Hardy became mentor of Indian mathematician Srinivasa Ramanujan and they came up with the Hardy-Ramanujan asymptotic formula which used for finding quantum partition functions of atomic nuclei. In 1930, he took part in the union of democratic control during World War I and got strongly involved in 'For Intellectual Liberty'.

He died on 1st December, 1947.

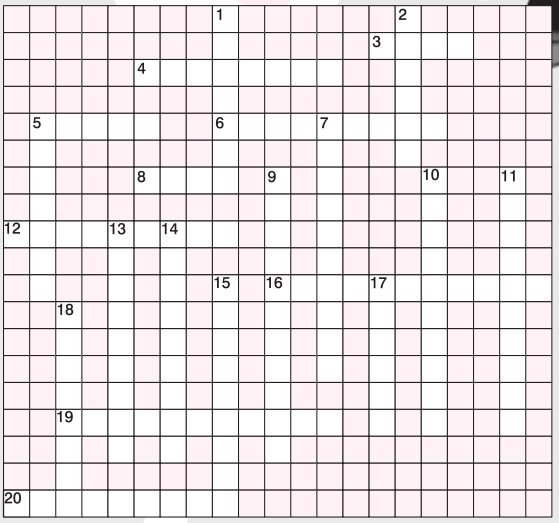
CROSSWORD

ACROSS

3. The vector sum of the three sides of a triangle taken in order is .... (4)
4. The mean of squares of the deviations from mean. (8)
5. The coefficients of the binomial expansions are arranged in ..... (5)
6. The triangle in which orthocentre, centroid, circumcentre and incentre lie on a straight line. (9)
8. ..... matrix is both symmetric and skew-symmetric. (4)
12. A function which is both injective and surjective. (9)
16. A chord through focus of conic section perpendicular to major axis. (11)
19. Sum of the product of elements of any row with their corresponding cofactors. (11)
20. Another word for x-coordinates. (9)

DOWN

1. If $|A| \neq 0$, then the homogeneous system $AX = 0$ has ..... solution. (7)
2. A point of intersection of a conic with its axis. (6)
5. The matrix formed by taking the transpose of the cofactor matrix of given matrix. (7)
7. Pair of circles who do not have radical axis. (10)
9. The hyperbola whose eccentricity is $\sqrt{2}$. (11)
10. Inverse of an even function is ..... . (10)
11. A set of events, if atleast one of the events must occur. (10)
13. If $(p + \sqrt{q})$ is one root of quadratic equation, then the other root is ..... of it. (9)
14. A pair of straight lines represented by equation $ax^2 + 2hxy + y^2 = 0$, where $h^2 < ab$. (9)
15. Lines lie in different planes. (9)
17. A line that intersects a curve at two points. (6)
18. The radius of circle that touches all sides of polygon. (8)

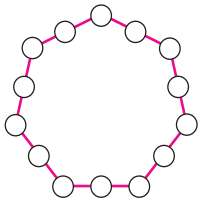


Puzzles to Puzzle You

1. The speedometer of the family car shows 15951 km. The driver noticed that this number is palindromic : it reads the same backward as forward.
"Curious" the driver said to himself, "It will be a long time before that happens again." But 2 h later, the speedometer showed a new palindromic number. How fast was the car travelling in those 2 h?

2. Two candles have different lengths and thicknesses. The long one can burn 7/2 h and the short one, 5 h.
After 2 h burning, the candles are equal in length. Two hours ago, what fraction of the long candle's height gave the short candle's height?

3. Can you arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 in the fourteen circles of adjoining figure so that the sum of three numbers on each side of heptagon is same?



4. A woman was carrying a basket of eggs to market when a passerby bumped her. She dropped the basket and all the eggs broke. The passerby, wishing to pay for her loss, asked "How many eggs were in your basket"?
"I don't remember exactly," the woman replied, "but I do recall that whether I divided the eggs by 2, 3, 4, 5 or 6, there was always 1 egg left over, when I took the eggs out in groups of 7, I emptied the basket."
What is the least number of eggs that broke?

5. Fill in the boxes with distinct integers 1 to 5 such that the following equation holds true :

$$\square + \square - \square \times \square \div \square = 6$$

6. Every morning Suresh walks to school. At one-fourth of the way he passes the petrol station and at one-third of the way, the railway station. At petrol station, its clock shows 7:30 am and at railway station, its clock shows 7:35 am. When does Suresh leave his house and when does he reach school?

7. Three brothers shared 24 apples, each getting a number equal to his age 3 yr before. The youngest one proposed a swap : "I will keep only half the apples I got and divide the rest between you two equally. But then the middle brother, keeping half his accumulated apples, must divide the rest equally between the oldest brother and me and then the oldest brother must do the same."
They agreed, the result was that each ended with 8 apples.
How old were the brothers?

SUDOKU

6		2			9			3
					2			4
	8			5	3		6	
7	2			4				
	1						3	
				2			4	1
	7		6	1			5	
4			2					
9			5			8		7

February 2017 issue Answer

Crossword

1 E	2 L	I	M	I	N	A	T	I	O	N			3 L	E	A	D	I	N	G	T	E	R	M
	A			4 O											5 O								6 P
	M			R											R								O
	I			D											T								I
	N			E											H								N
	A			7 R	H	O	M	B	O	I	8 D				O								T
				O							I				G								O
				F							R				O								F
				A							E				N								C
				M							C				A								O
				A							T				L								N
				T							R				M								T
				R							I				A								A
				I							X				T								C
				X											R								T
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														9 E	X	P	O	N	E	N	T		
10 F	A	C	T	O	11 R	T	R	E	E									12 C					
					H													Y					
					O													C					
					M													L					
					B													O					
					U													I					
					13 S	O	L	I	D	G	E	O	M	E	T	R	Y	D					

Puzzles to Puzzle you Answers February

1. 16/3 **2.** 84 years **3.** 630
4. 180 metres **5.** 2420 feet **6.** 4

SUDOKU

9	4	6	1	2	7	3	5	8
8	7	5	9	3	4	1	2	6
1	2	3	5	6	8	7	4	9
4	8	9	2	7	3	5	6	1
6	5	7	4	9	1	8	3	2
2	3	1	6	8	5	4	9	7
3	1	2	7	4	9	6	8	5
7	9	4	8	5	6	2	1	3
5	6	8	3	1	2	9	7	4

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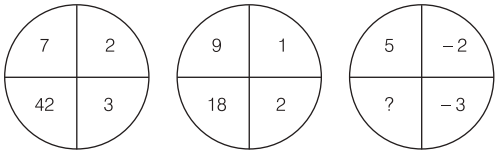
1. In a certain code language, 'nee muk pic' means 'grave and concern', 'ill dic so' means 'every body else' and 'tur muk so' means 'body and soul'. What will be the code for 'every concern'?
2. If the positions corresponding to the multiple of 5 in the English alphabet are replaced by symbols and that of multiples of 7 by digits, how many letters will be left?
3. Consider the following series :
   A B C D ... X Y Z | Y X ... B A | B C D ... Y Z |
   Y X ... C B A | B C ... Y Z | ...
   Which letter occupies the 1000th position in the above series?
4. Rohit walked 25 m towards South. Then, he turned to his left and walked 20 m. He then turned to his left and walked 25 m. He again turned to his right and walked 15 m. At what distance is he from the starting point and in which direction?
5. If 453945 stands for DECIDE, then what 8978 stands for?
6. In the following question, a set of figures carrying certain characters is given. Assuming that the characters in each set follow a similar pattern, find the missing character.



7. A student has to obtain 33% of the total marks to pass. He got 125 marks and failed by 40 marks. Find the maximum marks of this student.
8. Find the missing number.
   5, 10, 13, 26, 29, 58, 61, ?
9. A trader keeps the marked price of an item 35% above its cost price. Find the percentage of discount allowed to gain of 8%.
10. Find the missing figure in the given question.

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Write your answers on separate sheets and send to us along with this coupon

(ALL PARTICULARS TO BE FILLED UP IN BLOCK LETTERS)

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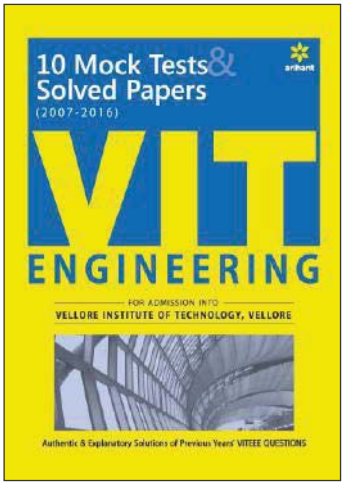

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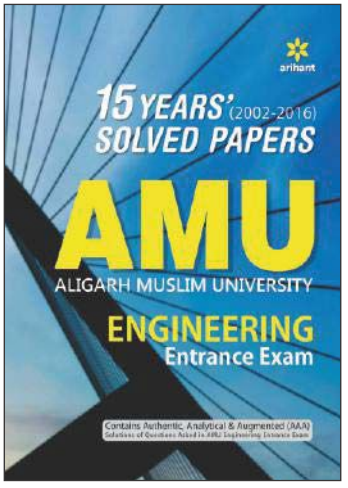

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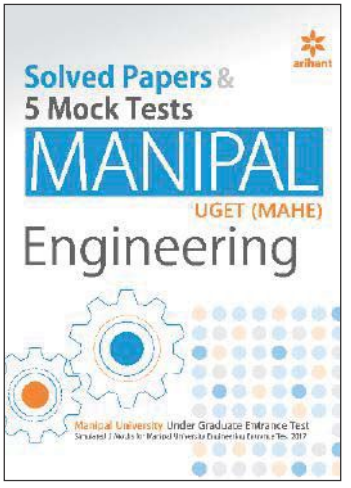

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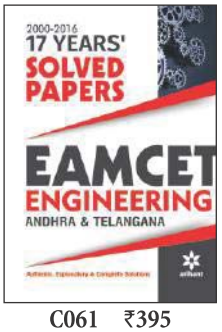

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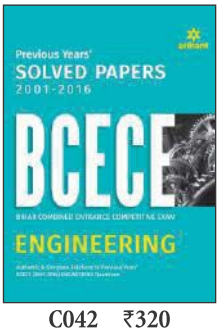

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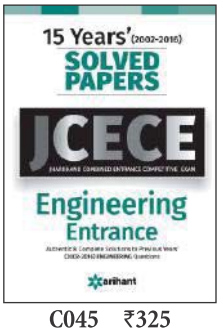

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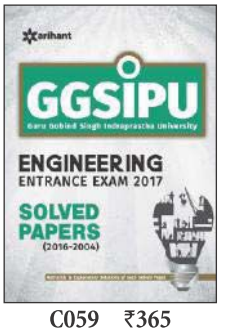

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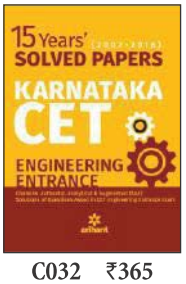

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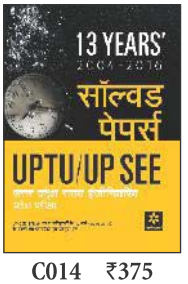

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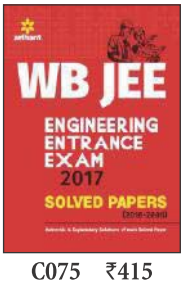

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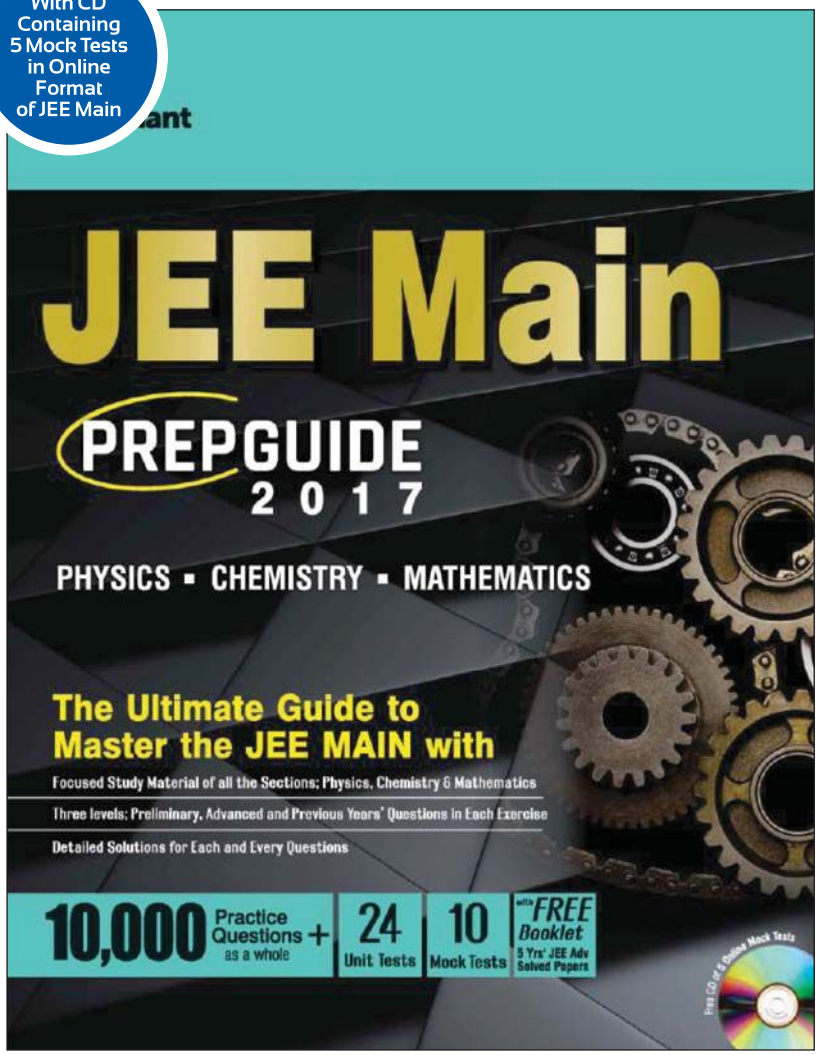
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